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A TREATISE

ON THE

APPLICATION OF ANALYSIS

TO

SOLID GEOMETRY.

τοῦ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν.

Πλάτων.

Duncan Farquharson

COMMENCED

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PREFACE.

THIS work was commenced by Mr. Gregory in the course of the year 1842; and in the Autumn of 1843 had received his final revision as far as the end of Chapter XI.: he had likewise made numerous extracts from various sources preparatory to writing Chapters XII., XIII., XIV., XV.: and had arranged a collection of Problems which, with slight additions, forms the subject of Chapter XVI., the last of the Treatise. His further progress in this work was unhappily arrested by death.

Having, in accordance with the last wishes of my most valued friend, undertaken the completion of this work, I have fulfilled my task to the best of my ability. It is hoped that the natural difficulty of bringing to a conclusion a treatise commenced by another, will secure for me the indulgence of the reader.

The principal object of this Treatise is to develop a system of Solid Geometry, in a form suitable to mathematical students, by means of symmetrical equations. The general advantage of symmetry in this branch of mathematics is so striking, that the utility of such a work will be at once recognized. There

are undoubtedly many cases in which unsymmetrical methods have the advantage of brevity; a bigoted adherence to symmetrical investigations has therefore been avoided.

It is scarcely necessary for me to state that I have derived great assistance from Leroy's *Géométrie des Trois Dimensions*, Moigno's *Calcul Différentiel*, Gregory's *Examples of the Processes of the Differential and Integral Calculus*, and from several articles in the *Cambridge Mathematical Journal*. My numerous obligations to my mathematical friends have been acknowledged in the course of the work.

WILLIAM WALTON, M.A.
Trinity College.

CAMBRIDGE, *January* 1845.

CONTENTS.

CHAPTER I.

Exposition of Principles and Fundamental Theorems.

CHAPTER II.

Of the Straight Line and Plane.

CHAPTER III.

Transformation of Co-ordinates.

CHAPTER IV.

Reduction of the General Equation of the Second Degree.

CHAPTER V.

Interpretation of the Equation of the Second Degree.

CHAPTER VI.

Theorems relating to Surfaces of the Second Order.

CHAPTER VII.

Of Curves in Space.

CHAPTER VIII.

Of Tangents and Normals to Surfaces.

CHAPTER IX.

Of Tangents to Curves, Normal and Osculating Planes.

CHAPTER X.

Of the Generation of Surfaces by the Motion of Curves.

CHAPTER XI.

Envelops to Surfaces.

CHAPTER XII.

On Partial Differential Equations to Families of Surfaces.

CHAPTER XIII.

On Singular Points and Lines of Surfaces.

CHAPTER XIV.

On the Curvature of Curves in Space.

CHAPTER XV.

On the Curvature of Surfaces.

ERRATA.

PAGE	LINE	CORRECTIONS.
1,	*Note.*	Not any remarks on the subject of this Note have been found among Mr. Gregory's MSS.
2,	10 from bottom.	*Read* Oz *instead of* Ox. vice versâ
16,	5 from bottom.	The figure ought to be so drawn that M_2 shall lie within the area OMM_1.
31,	13.	*Read* r being the ratio of the distance between the points xyz, $x_1y_1z_1$, to that between the points $x_1y_1z_1$, $x_2y_2z_2$; *instead of* r being ..... (x_1, y_1, z_1).
35,	8 from bottom.	*Read* $-y_1z_3$ *instead of* $-y_3z_1$.
40,	13.	*For* $(\alpha \sim \alpha')\lambda + (\beta \sim \beta')\mu + (\gamma \sim \gamma')\nu$, *Read* $\pm\{(\alpha - \alpha')\lambda + (\beta - \beta')\mu + (\gamma - \gamma')\nu\}$.
40,	15.	*Instead of the expression for* δ, take $\delta = \pm \dfrac{(mn' - m'n)(\alpha - \alpha') + (nl' - n'l)(\beta - \beta') + (lm' - l'm)(\gamma - \gamma')}{\sin\theta}$.
66,	4 from bottom.	*Read* $P''\gamma + Q''$ *instead of* $P\alpha + Q$.
74,	11.	*For* (y, z) *read* (x, y).
92,	15, 17, 18.	For r and r', wherever they occur, read r_1 and r_1'.
98,	9.	*For* condition *read* function.
98,	11 from bottom.	*By* this cutting plane, *is meant* a plane of parabolic section, *which in this Article is proved to be parallel to one, and to one only, of the generating lines of the asymptotic cone.*
125,	13.	$\cos\theta$ *should be replaced by* $\sin\theta$: *corrections of consequent errors must be made throughout* Art. (152).
137,	4.	*Insert* $=0$ *just before the comma.*
149,	9.	There are errors in Art. (181) consequent on those in Art. (152).
161,	7.	*For* α, β, γ, *read* x, y, z.

APPLICATION OF ANALYSIS

TO

SOLID GEOMETRY.

CHAPTER I.

EXPOSITION OF PRINCIPLES, AND FUNDAMENTAL THEOREMS.

Elementary Notions.

ART. (1). It is necessary for the Application of Analysis to Geometry that we should have the means of expressing by symbols, not only the absolute magnitudes of geometrical quantities, such as lines, areas, angles, &c., but also the positions of points. Our habit of denoting arithmetical quantities by a single symbol naturally leads us also to denote the simplest geometrical magnitudes by a single symbol, and thus we represent straight lines of different lengths by such symbols as $a, b, x, y, \alpha, \beta$. Then, in virtue of a principle we shall here assume as known, an area is denoted by the product of two, and a solid by that of three such symbols *considered as numbers.** Angles, being a species of geometrical magnitude not homogeneous with straight lines, we shall denote also by single letters, using generally the Greek letters $\lambda, \mu, \nu, \theta, \phi, \psi$. Functions of angles, such as sines, cosines, &c., we shall often denote also by single letters: they may always be considered as the ratios of two of the symbols of straight lines.

* The reason of this symbolization will be found in the Appendix.

(2) The position of a point in space is determined by referring it to three fixed lines intersecting each other in one point. And the mode by which this is done is the characteristic feature of the Application of Analysis to Geometry.

Any fixed line is called an *axis*, and the three fixed intersecting lines are called the *co-ordinate axes*, their point of intersection being named the *origin*. Each of these lines may be considered as determined by the intersection two and two of three planes. Thus, in fig. (1), if Ox, Oy, Oz be the three co-ordinate axes, O the origin, we may consider the axis Ox as the intersection of the planes zOx and yOx; the axis Oy as the intersection of zOy and xOy, and the axis Oz as the intersection of xOz and yOz. These planes are termed the *co-ordinate planes*, and may be used as fixed planes to which the position of a point in space may be referred. In speaking of these planes we shall call xOy the plane xy, yOz the plane yz, and xOz the plane xz.

(3) To shew how a system of co-ordinate axes may be used for determining the position of a point in space, let P (fig. 1) be a point situate within the solid angle $Oxyz$, and through P draw PA, PB, PC parallel to the three co-ordinate axes Ox, Oy, Oz respectively, and meeting the co-ordinate planes in A, B, C; then the position of the point P is known if we know the lengths of the lines PA, PB, and PC, which are called the *co-ordinates* of P. For if along the line Ox we measure a length OD equal to PA, and through D draw a plane parallel to the plane yOz, every point in this plane has a line equal to OD or PA as its co-ordinate parallel to Oz. In like manner, if we measure along Oy and Oz lengths OE and OF equal to PB and PC respectively, and through E and F draw planes parallel to zOx and xOy, every point in the former has its co-ordinate parallel to Oy equal to OE or PB, and every point in the latter has its co-ordinate parallel to Oz equal to OF or PC. Hence, the point which is determined by the intersection of these three planes has for its co-ordinates parallel to Ox, Oy, Oz, the lines PA, PB, PC respectively. In other words, the position of the point P is determined by the preceding construction, and there-

fore the position of a point may be considered as known when the lengths of its co-ordinates are given.

It is easy to see from the figure that the intersections of the three planes drawn through D, E, and F, with each other and with the co-ordinate planes, determine a parallelopiped, of which O and P are opposite solid angles. Hence we may obtain the point P by a simpler construction; for, since $OD = PA$, $DC = OE = PB$, if along Ox we measure $OD = PA$, and at D draw DC parallel to Oy and equal to PB, and through C draw CP parallel to Oz, making it of the given length, the point P will be determined. We might of course equally well begin by measuring the first co-ordinate along either of the other axes.

The co-ordinates PA, PB, PC of a point P being different lengths of straight lines are, according to the explanation in Art. (1), usually represented by the symbols x, y, z, when they are indeterminate, and by other letters, as a, b, c, or α, β, γ, when determinate values are assigned to them.

(4) In what precedes we assumed that the point P is within the solid angle $Oxyz$, and that the co-ordinates x, y, z are measured along Ox, Oy, and Oz in one direction only: we have therefore as yet the means of determining the position of a point only within a limited portion of space. For, since the lines which intersect at O may be considered as infinite in length, the three planes, which by their intersection determine these lines, divide space into eight solid angles, of which we have considered but one. If we indicate by x', y', z' arbitrary points in the prolongation of the axes, these eight solid angles may be denoted by

$$\begin{array}{cccc} Oxyz, & Ox'yz, & Oxy'z, & Oxyz', \\ Oxy'z', & Ox'yz', & Ox'y'z, & Ox'y'z'; \end{array}$$

and for each of these divisions we should require to use a separate set of symbols to indicate in which octant the point under consideration is situate, so that eight sets of formulæ would be required in discussing the position of a point in all possible positions. The artifices of analysis fortunately enable us to avoid this complexity by reducing all these sets to

one; and this is done by the aid of the algebraical symbols $+$ and $-$, in the following manner.

(5) We agree, as may be done consistently with the properties of the symbols, that, when starting from a given point a straight line of given length is considered as positive, a line of the same length measured in the opposite direction is to be reckoned as negative. Then, if lines measured from O towards x (fig. 2) be positive, those measured from O towards x' are negative; and if lines measured from O towards y be positive, those measured from O towards y' are negative; and if those measured from O towards z be positive, those from O towards z' are negative. Now the co-ordinates of a point P in the octant $Oxyz$ are measured along Ox, Oy, Oz, and are therefore by agreement all positive. But the co-ordinates of a point P' in the octant $Oxyz'$ are measured along Ox, Oy, and Oz'; consequently the first two are positive and the third negative. Hence, any formula involving the co-ordinates of P may be transformed into one involving those of P' simply by putting $-z$ for z, or changing the sign of z. In like manner, if we have a point P'' in the octant $Ox'y'z$, its co-ordinates are measured along Ox', Oy', Oz, consequently the first two are negative and the third positive; so that a formula involving the co-ordinates of P may be transformed into one involving those of P'' by changing the sign of x and of y. In a similar manner we may proceed for all the octants according to the following scheme:—

In the octant	$Oxyz$	the co-ordinates are	$+x, +y, +z,$
....	$Ox'yz$	....	$-x, +y, +z,$
....	$Oxy'z$	....	$+x, -y, +z,$
....	$Oxyz'$	....	$+x, +y, -z,$
....	$Oxy'z'$	....	$+x, -y, -z,$
....	$Ox'yz'$	....	$-x, +y, -z,$
....	$Ox'y'z$	....	$-x, -y, +z,$
....	$Ox'y'z'$	....	$-x, -y, -z.$

It appears then that by supposing each of the quantities x, y, z to be both absolutely positive and absolutely negative, and by combining these in all possible ways, we can represent

the position of a point in any octant, that is, in any part of space.

(6) In defining the co-ordinate axes we made no restrictions as to the angles at which they are inclined to each other, but it is usually most convenient to use as co-ordinate axes three straight lines which are at right angles to each other: such a system is called a system of *rectangular co-ordinates*.

Interpretation of Equations.

(7) The results of the applications of Analysis to Geometry are expressed in equations involving the co-ordinates which have been denoted by x, y, z; we must therefore, before proceeding further, consider what is the geometrical interpretation of such equations. Let us take a single equation, such as

$$f(x, y, z) = 0:$$

this may be considered as a relation which enables us to determine any one of the variables when the other two are given, two being always arbitrary. Let these be x and y, so that the equation is equivalent to another of the form

$$z = \phi(x, y);$$

then we are at liberty to assign arbitrary and independent values to x and y, and for every such pair we obtain from the equation a definite value for z. Now to every pair of values of x and y there corresponds a point in the plane of xy; and if through this we draw a line parallel to the axis of z, and measure along it a length equal to the value of z given by the equation, it is clear that we shall in that way obtain a series of points constituting a *surface*, not forming a solid, since we take only one point in each co-ordinate parallel to the axis of z, which is drawn through every point in the plane of xy. We here suppose that the equation $z = \phi(x, y)$

gives only one value of z for each pair of values of x and y; but if it should give several values, the only difference is that in each co-ordinate parallel to z we must take a determinate number of points, and these taken together will constitute a surface of several sheets.

It is to be remarked, that though we spoke of assigning arbitrary values to x and y, they must be such as will give only *possible* values to z; that is to say, will affect it with the signs + and − only, for we confine our interpretations to such results. If the equation cannot be satisfied by combinations of possible values of the variables, its interpretation does not come within the scope of our present purpose. Should however it be possible to satisfy the equation by dividing it into a system of two or three other simultaneous equations, it will then represent a limited number of lines or of points, according to a principle of which we shall speak immediately. Thus the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=0,$$

which is satisfied by no possible values of the variables, except

$$x=a, \quad y=b, \quad z=c,$$

represents a point. If the equation be satisfied by several distinct independent equations, it represents as many distinct surfaces.

(8) If the equation involve only two out of three of the variables, it still represents a surface, but one of peculiar kind. Thus, if we have the equation

$$f(x, y)=0,$$

it is satisfied by certain values of x and y, independently of z. Here x and y are no longer both arbitrary, but one is given in terms of the other by the equation; to each pair corresponds a point in the plane of xy, and the series of such points constitutes a curve in that plane. If through each point in this curve we draw a co-ordinate parallel to z, every point in that co-ordinate has the same values of x and y as its co-ordinates parallel to these axes; and therefore the equation

$$f(x, y)=0$$

is true for every point along each co-ordinate parallel to z drawn through each point in the curve. That is to say, the equation

$$f(x, y)=0$$

represents a surface such that every straight line drawn parallel to z through a certain series of points in the plane of xy lies wholly in the surface. Such surfaces are called cylindrical, the

common right cylinder having been the first of the kind of which the properties were known.

If the equation contain only one of the variables, so that it is of the form

$$f(x) = 0,$$

it can always, by the theory of equations, be decomposed into simple factors of the form $x - a$. If the second term of this factor, or a, be a possible quantity, the equation

$$x - a = 0$$

indicates a series of points of which the co-ordinates parallel to x are equal, that is, a plane parallel to yz: if the second term be not possible we do not interpret the equation. Hence, the equation

$$f(x) = 0$$

represents as many planes parallel to yz as it contains possible linear factors of the form $x - a$. Thus we see that in all cases when a single equation is interpreted, it represents a surface of some kind or other.

(9) When two simultaneous equations are given, as

$$f(x, y, z) = 0, \quad f_1(x, y, z) = 0,$$

each of these represents a surface, and when they are combined the co-ordinates x,y,z must belong to points common to the two surfaces, that is to say, to the line of intersection of the two surfaces. Hence, two simultaneous equations represent a line which will be in general a curve of double curvature, unless either one of the equations be that to a plane, or the combination of the two lead to the equation to a plane. Since two equations may be combined in an infinite number of ways, the result of any such combination is the equation to some surface which passes through the intersection of the two given surfaces. Any such result may be used instead of one of the given equations, if such a change conduce to simplicity. Thus, if we combine the equations so as to eliminate any one of the variables, the resulting equation may be used instead of one of the given equations. Suppose that z is the variable which is eliminated, so that

$$\phi(x, y) = 0$$

is the resulting equation. This, by Art. (8), is the equation to

a cylindrical surface parallel to the axis of z; and as we may obtain similar equations for each of the other axes, it appears that any line in space may be considered as the intersection of two cylindrical surfaces parallel to two of the co-ordinate axes.

(10) If we wish to determine the curve in which a surface is cut by one of the co-ordinate planes, as that of xy for instance, we must combine the equation to the surface

$$f(x, y, z) = 0 \quad \text{with } z = 0,$$

as for all points in the plane of xy the co-ordinate z is zero: these two equations taken together determine the curve of intersection, or, as it is called, the *trace* of the surface on the plane of xy. Even though the equation do not contain z, it must be combined with the equation

$$z = 0;$$

since, when taken by itself, an equation of the form

$$f(x, y) = 0$$

represents a cylindrical surface as we have just seen.

If, instead of supposing $z = 0$, we combine

$$f(x, y, z) = 0 \quad \text{with } z = a,$$

we determine the intersection of the surface with a plane of which every point is at the same distance from the plane of xy, that is, which is parallel to it. The substitution of a for z in the equation to the surface gives

$$f(x, y, a) = 0,$$

which, considered by itself, is a cylindrical surface, and when combined with $z = 0$ it gives us the trace of the cylinder on the plane of xy, which is clearly the same curve as the intersection of

$$f(x, y, z) = 0 \quad \text{with } z = a.$$

Of the intersection of a surface by other planes we shall speak elsewhere.

(11) When three simultaneous equations are given, it is easy to see that they are sufficient for determining absolutely the values of the three variables x, y, z, and consequently that they must represent one or more points.

Fundamental Theorems.

(12) *Theory of Projections.* When a point is referred to a plane by means of a straight line drawn parallel to a fixed axis, the point where the line meets the plane is called the *projection* of the point on the plane. Thus in fig. (1) A is the projection of P on the plane of yz, B is the projection on the plane of xz, and C that on xy. If a series of points, forming a line, be in this way projected on any plane, their projections constitute a line which is called the projection of the line on the plane.

When one line or several lines connected together enclose a plane area, the area enclosed by the projection of the lines is called the projection of the first area. If the plane on which the projection is made be perpendicular to the fixed axis, the projection is called orthogonal, and it is this kind which we shall have chiefly to consider: unless, therefore, the contrary be expressly stated, the projection is always to be considered as orthogonal.

This idea of projection may, in the case of the straight line, be somewhat extended; for if from the extremities of any terminated straight line we draw perpendiculars to a line fixed in position, the portion of the latter intercepted between the feet of the perpendiculars is also called the projection of the former line on the fixed line.

From this definition, combined with what has been said in Art. (3), it is easy to see that the rectangular co-ordinates of a point are the orthogonal projections on the co-ordinate axes of its distance from the origin.

(13) The general property of all orthogonal projections of bounded straight lines or plane areas, is that the projections are equal to the original line or area multiplied by the cosine of the angle between the straight line or plane area, and that on which it is projected. This must be proved separately in each case.

1st, *When a straight line is projected on a plane.* Let PQ (fig. 3) be the given terminated straight line, $ABCD$ the plane of projection: draw PM, QN perpendicular to it; then MN is by definition the projection of PQ on $ABCD$. Since PM and

QN are both perpendicular to the same plane, they are parallel to each other; in the plane therefore in which they lie draw PR parallel to MN, and meeting QN in R, so that PR is equal to MN. Now the inclination of a straight line to a plane is the angle which the line makes with the intersection of the plane and a plane perpendicular to it passing through the line. Since, then, PM and QN are perpendicular to $ABCD$, the plane of $PQMN$ is also perpendicular to it, and the inclination of PQ to the plane $ABCD$ is measured by the angle between PQ and MN or the equal angle QPR. Let this be θ, then in the triangle PQR

$$PR = PQ \cos \theta,$$

and therefore $$MN = PQ \cos \theta,$$

as was to be proved.

It is to be observed that we consider the inclination of the straight line to the plane to be the *acute* angle which it makes with its projection.

2nd, *When a straight line is projected on another straight line.* Let PQ (fig. 4) be the terminated straight line, AB the line on which it is to be projected, and which is not necessarily in the same plane with PQ. In such a case, since the lines do not meet, their inclination is measured by the angle between one of them, and a parallel to the other drawn through any point in it. Draw PM, QN perpendicular to AB; then, by definition, MN is the projection of PQ on AB. Through QN draw a plane perpendicular to AB, and let R be the point where it is met by a parallel to AB drawn through P: join QR. Then, since a straight line which is perpendicular to a plane is perpendicular to every straight line in the plane, the angle PRQ is a right angle; and therefore

$$PR = PQ \cos QPR.$$

But since $PRMN$ is a rectangle, $PR = MN$; so that, calling θ the inclination of PQ to AB, we have

$$MN = PQ \cos \theta,$$

as was to be proved. The angle θ is, as before, supposed to be the *acute angle* which PQ makes with AB.

If instead of two fixed points PQ, connected by a straight line, we have any number of points $PQ\,P_1Q_1$ (fig. 5) connected by straight lines PQ, QP_1, P_1Q_1, and if from P, Q, P_1, Q_1 we draw on AB the perpendiculars PM, QN, P_1M_1, Q_1N_1, the whole line MN_1 is composed of the projections MN_1, NM_1, M_1N_1. But MN_1 may be considered as the projection of a single line PQ_1 connecting P and Q_1; therefore the projection of any single line connecting two points is equal to the sum of the separate projections of any number of connected lines which join the same points. Such a series of lines may be called a broken line; and we may thus say generally that if any two points be connected by a straight line or by any series of broken lines, their projections on any line are equal. This is a proposition which we shall frequently have occasion to use.

In the figure we have supposed all the separate projections to be additive; but if one of the points, as Q_1, were in the position Q_2, the projection of P_1Q_2 must be subtracted: we may, however, get rid of the necessity of attending to this. For if we consider the angles as measured by the inclination of lines estimated all in the same direction, as for instance the inclination of PQ to AB and of P_1Q_2 to AB and not to BA, it is clear that the latter will be an obtuse angle whenever by the position of Q_2 the projection of P_1Q_2 is to be subtracted; and hence the sign of the term is given by the sign of the cosine.

3rd, *When a plane area is projected on a plane.* We shall begin with a triangle of which one side is parallel to the plane of projection. Let ABC (fig. 6) be the triangle, $A'B'C'$ its projection, of which we suppose the side $B'C'$ to be parallel to BC. Through AA' draw a plane perpendicular to BC and $B'C'$, which therefore cuts the triangle and its projection in the lines AD and $A'D'$ perpendicular to BC and $B'C'$. The area of the triangle ABC is then equal to $\frac{1}{2}BC.AD$, while that of $A'B'C'$ is equal to $\frac{1}{2}B'C'.A'D'$ or $\frac{1}{2}BC.A'D'$. But $A'D'$, being the projection of AD on the plane, is equal to $AD \cos\theta$, if θ be the inclination of the plane of the triangle to the plane of projection, or of AD to $A'D'$. Hence we have

$$A'B'C' = ABC \cos\theta.$$

If one of the sides of the triangle be not parallel to the plane of projection, we may draw through one angle, as B, (fig. 6a) a plane parallel to the plane of projection and meeting the plane of the triangle in some line BD. Then by what has preceded, as $A'BD$ is the projection of ABD, $BC'D$ of BCD and $A'BC'$ of ABC, we shall have

$$A'BD = ABD \cos\theta, \quad BC'D = BCD \cos\theta;$$

since the side BD of each of these triangles is parallel to the plane of projection. Hence, subtracting the latter from the former,

$$A'BD - BC'D = (ABD - BCD) \cos\theta,$$

or

$$A'BC' = ABC \cos\theta,$$

as was to be proved.

Since every polygon may be divided into a number of triangles, of each of which the preceding proposition is true, it applies also to the sum of the triangles, that is, to the polygon. Also, the theorem may, by the method of limits, be extended to curvilinear areas, since they may always be considered as the limits of polygons of which the proposition is true.

By means exactly similar to those employed in the case of several series of lines terminated at the same point, we may shew that if any number of straight lines be connected either by one plane area of which they are the boundaries, or by any number of plane areas having common edges, the projections of all on any plane are equal.

(14) *To express the distance between any two points in terms of their rectangular co-ordinates.*

Let PQ (fig. 7) be the two points, and assume Ox, Oy, Oz as rectangular co-ordinates; draw PN, QN' parallel to Oz, and NM, $N'M'$ parallel to Oy; then OM, MN, NP are the co-ordinates of P and OM', $M'N'$ and $N'Q$ are those of Q. Let

$$OM = x, \quad MN = y, \quad NP = z, \quad OM' = x_1, \quad M'N' = y_1, \quad N'Q = z_1;$$

let $PQ = r$, and let λ, μ, ν be the angles which it makes with axes of x, y, and z. On PQ, as a diagonal, construct a rectangular parallelopiped, the sides of which parallel to the axes of x, y, z are equal to

$$x_1 - x, \quad y_1 - y, \quad z_1 - z.$$

Now if we project on PQ the broken line $PTSQ$, we have

$$r = (x_1 - x) \cos \lambda + (y_1 - y) \cos \mu + (z_1 - z) \cos \nu.$$

Again, projecting PQ on the axes of x, y, and z, we have

$$x_1 - x = r \cos \lambda, \quad y_1 - y = r \cos \mu, \quad z_1 - z = r \cos \nu;$$

multiplying these equations by $x_1 - x$, $y_1 - y$, $z_1 - z$ respectively, and adding, we have, by the previous equation,

$$r^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2,$$

which is the required expression.

If the point Q be the origin, then $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, and we have for the distance of P from the origin,

$$OP = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

(15) *To express the distance between two points in terms of their oblique co-ordinates.*

Make a construction similar to that in the last article, and let α, β, γ be the angles between Oy and Oz, Ox and Oz, Ox and Oy: then, using the same notation as before, and projecting the broken line $PTSQ$ on PQ, we have

$$r = (x_1 - x) \cos \lambda + (y_1 - y) \cos \mu + (z_1 - z) \cos \nu.$$

Again, projecting PQ and the broken line $PTSQ$ on the axis of x, the two projections are equal because PQ and $PTSQ$ have the same extremities; hence

$$r \cos \lambda = x_1 - x + (y_1 - y) \cos \gamma + (z_1 - z) \cos \beta,$$

as γ is the angle between Ox and Oy, and β that between Ox and Oz. Similarly for the other co-ordinate axes we have

$$r \cos \mu = y_1 - y + (z_1 - z) \cos \alpha + (x_1 - x) \cos \gamma$$
$$r \cos \nu = z_1 - z + (x_1 - x) \cos \beta + (y_1 - y) \cos \alpha.$$

Multiplying these equations by $x_1 - x$, $y_1 - y$, $z_1 - z$ respectively, and adding, we have, in consequence of the preceding equation,

$$r^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 + 2(x_1 - x)(y_1 - y) \cos \gamma$$
$$+ 2(x_1 - x)(z_1 - z) \cos \beta + 2(y_1 - y)(z_1 - z) \cos \alpha.$$

It is obvious that this gives us the expression for the length of a diagonal of a parallelopiped in terms of the sides and the angles which they make with each other.

And if as bef. $x_1 = y_1 = z_1 = 0$. or Q be origin

$r^2 = x^2 + y^2 + z^2 + 2xy \cos \gamma + 2xz \cos \beta +$ [illegible]

(16) *To find the relation between the cosines of the angles which a straight line makes with three rectangular axes.*

Taking the origin O (fig. 8) in the line, let $POx = \alpha$, $POy = \beta$, $POz = \gamma$, and let x, y, z be the co-ordinates of any point P in the line; then if the distance OP be r, we have, by Art. (14),

$$r^2 = x^2 + y^2 + z^2;$$

but since x, y, z are the projections of r on the co-ordinate axes, we have $x = r \cos \alpha, \quad y = r \cos \beta \quad z = r \cos \gamma;$

therefore, substituting for these quantities, we have

$$r^2 = r^2 (\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

or $$\cos^2\alpha + \cos^2\beta + \cos^2 \gamma = 1,$$

a very important relation, to which we shall frequently refer. The cosines of the angles which a straight line makes with the co-ordinate axes are quantities which we shall often have occasion to use, and as they serve to determine the *direction* of the line, we shall call them the *direction-cosines* of the line; and when we wish to speak of a straight line with reference to its direction-cosines, which we may call l, m, n, we shall name it the line $[l, m, n]$.

(17) The preceding theorem enables us to prove a very important property of the orthogonal projections of plane areas. For since any two planes make with each other the same angle as two lines respectively perpendicular to them, if we have a plane area perpendicular to the line of which the direction-cosines are $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, its inclinations to the co-ordinate planes of yz, zx, xy are α, β, γ respectively. Therefore if the magnitude of the area be denoted by A, and those of its projections on yz, zx, and xy by A_x, A_y, A_z, we shall have

$$A_x = A \cos \alpha, \quad A_y = A \cos \beta, \quad A_z = A \cos \gamma.$$

Squaring and adding these, and observing that by the preceding theorem $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

we have $$A^2 = A_x^2 + A_y^2 + A_z^2;$$

or the square of any plane area is equal to the sum of the squares of its projections on three planes at right angles to each other.

(18) *To express the cosine of the angle between two straight lines in terms of the direction-cosines of the lines.*

If the lines do not meet, the angle between them is found by drawing through any point in the one a line parallel to the other. Take this point as the origin O (fig. 9); let α, β, γ be the angles which OP, and $\alpha_1, \beta_1, \gamma_1$ those which OQ makes with the rectangular axes. Take in OP any point P of which the co-ordinates are x, y, z, and in OQ any point Q of which the co-ordinates are x_1, y_1, z_1, and join PQ; let $OP = r$, $OQ = r_1$, $PQ = \delta$, and POQ (the angle between the lines) $= \theta$. Then, in the triangle POQ, we have

$$\delta^2 = r^2 + r_1^2 - 2rr_1 \cos\theta;$$

but if we express δ in terms of the co-ordinates of its extremities we have, by Art. (14),

$$\begin{aligned}\delta^2 &= (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 \\ &= x_1^2 + y_1^2 + z_1^2 + x^2 + y^2 + z^2 - 2(xx_1 + yy_1 + zz_1) \\ &= r^2 + r_1^2 - 2(xx_1 + yy_1 + zz_1).\end{aligned}$$

Equating these two values of δ^2, we have

$$rr_1 \cos\theta = xx_1 + yy_1 + zz_1;$$

but $\quad x = r\cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\gamma,$

$$x_1 = r_1\cos\alpha_1,\ \ y_1 = r_1\cos\beta_1,\ \ z_1 = r_1\cos\gamma_1,$$

therefore

$$\cos\theta = \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1.$$

(19) This theorem proves the following proposition: the projection of any finite straight line on another may be found by first projecting the line on three rectangular axes, and then taking the sum of these projections projected on the second line. For if r be the length of the finite line, and α, β, γ the angles which it makes with the axes, its projections on them are

$$r\cos\alpha, \quad r\cos\beta, \quad r\cos\gamma.$$

Then if $\alpha_1, \beta_1, \gamma_1$ be the angles which the second line makes with the axes, the projections of the preceding quantities on the second line are

$$r\cos\alpha\cos\alpha_1, \quad r\cos\beta\cos\beta_1, \quad r\cos\gamma\cos\gamma_1,$$

and their sum is

$$r(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1) = r\cos\theta,$$

θ being the angle between the lines.

The same proposition is applicable to a plane area; so that to find the projection of any plane area on a plane it is sufficient to project it on the three co-ordinate planes, and then to take the sum of these projections projected on the second plane.

(20) *To express the area of a triangle in terms of the co-ordinates of its angular points.*

This may be most conveniently done by first finding the projection of the area of a triangle, one of whose angular points is at the origin. Let AOB (fig. 10) be such a triangle, and MOM_1 its projection; then if $OM = r$, $OM_1 = r_1$, $MOx = \theta$, $M_1Ox = \theta_1$,

$$\text{area } MOM_1 = \tfrac{1}{2} rr_1 \sin(\theta - \theta_1).$$

Let the co-ordinates of M be x, y; and of M_1, x_1, y_1; then

$$\text{area } MOM_1 = \tfrac{1}{2}(x_1y - xy_1),$$

which is the expression for the projection of AOB on the plane of xy.

Now let ABC be the triangle of which the area is to be determined: x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, the co-ordinates of A, B, C; join its angular points with the origin, so as to form the three triangles AOB, BOC, COA. Then, by what has preceded,

$$\begin{aligned} \text{projection on } xy \text{ of } AOB &= \tfrac{1}{2}(x_1y - xy_1) \\ \ldots\ldots\ldots\ldots \text{ of } BOC &= \tfrac{1}{2}(x_1y_2 - x_2y_1) \\ \ldots\ldots\ldots\ldots \text{ of } COA &= \tfrac{1}{2}(x_2y - xy_2) \end{aligned}$$

But, by Art. (13. 3), since the triangle ABC, and the three triangles AOB, BOC, COA, are terminated by the same lines AB, BC, AC, the projection MM_1M_2 of ABC on xy is equal to the sum of the projections of the other triangles on that plane.

Hence, calling that projection A_z, we have

$$A_z = \tfrac{1}{2}(x_1y - xy_1 + x_2y_1 - x_1y_2 + xy_2 - x_2y),$$

since it is clear from the figure that the projections of BOC and COA must be subtracted; and in like manner, if A_y, A_x be the projections of ABC on the planes of xz and yz, we have

$$A_y = \tfrac{1}{2}(xz_1 - x_1z + x_2z - xz_2 + x_1z_2 - x_2z_1),$$

$$A_x = \tfrac{1}{2}(zy_1 - z_1y + yz_2 - y_2z + y_2z_1 - y_1z_2).$$

But since the co-ordinates are rectangular, we have (Art. 17), if A be the area of the triangle ABC,

$$A^2 = A_x^2 + A_y^2 + A_z^2,$$

and thus the area is expressed in terms of the co-ordinates of the angular points of the triangle.

From the nature of projections it appears that the cosines of the angles which the plane of ABC makes with the co-ordinate planes of xy, xz, and yz, are

$$\frac{A_z}{A}, \ \frac{A_y}{A}, \ \frac{A_x}{A} \quad \text{respectively.}$$

(21) *To express the volume of a tetrahedron in terms of the co-ordinates of its angular points.*

Take for simplicity one of the angular points as origin, and let $OABC$ (fig. 10) be the tetrahedron; then if OH be drawn perpendicular to the plane of ABC, and be put equal to h, the volume of the tetrahedron $V = \frac{1}{3}Ah$, A being, as in the last article, the area of ABC.

Now since OH is perpendicular to the plane of ABC, it makes with the co-ordinate axes of x, y, z the same angles that the plane of ABC makes with the planes of yz, xz, and xy; hence, by the last article, if α, β, γ be these angles,

$$\cos\alpha = \frac{A_x}{A}, \ \cos\beta = \frac{A_y}{A}, \ \cos\gamma = \frac{A_z}{A}.$$

But if we project the broken line $ONMA$ on OH, the sum of the projections is equal to OH, since OH, being perpendicular to the plane of ABC, is perpendicular to every straight line in it, and therefore to AH. Hence we have

$$h = \frac{A_x}{A}x + \frac{A_y}{A}y + \frac{A_z}{A}z;$$

therefore substituting for A_x, A_y, A_z their values previously found, and cancelling the terms which destroy each other, we find $\quad V = \frac{1}{6}(xz_1y_2 - xy_1z_2 + x_1yz_2 - x_2yz_1 + x_2zy_1 - x_1zy_2)$.

If we wish to introduce the co-ordinates x_3, y_3, z_3 of the fourth point, we have merely to substitute $x - x_3$, $y - y_3$, $z - z_3$, $x_1 - x_3$, $y_1 - y_3$, $z_1 - z_3$, $x_2 - x_3$, $y_2 - y_3$, $z_2 - z_3$ for the simple co-ordinates.

(22) As we shall have frequent occasions to use several analytical theorems which are perhaps not generally known, it will be of advantage to premise them here.

I. If
$$\frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c.$$
each of these ratios is equal to
$$\frac{(a^2 + a_1^2 + a_2^2 + \&c.)^{\frac{1}{2}}}{(b^2 + b_1^2 + b_2^2 + \&c.)^{\frac{1}{2}}},$$
and to
$$\frac{na + n_1a_1 + n_2a_2 + \&c.}{nb + n_1b_1 + n_2b_2 + \&c.};$$
n, n_1, n_2, &c. being any quantities whatever.

For assuming each of the ratios equal to r, we have
$$a = rb, \quad a_1 = rb_1, \quad a_2 = rb_2, \ \&c.$$
Squaring and adding,
$$a^2 + a_1^2 + a_2^2 + \&c. = r^2(b^2 + b_1^2 + b_2^2 + \&c.);$$
whence, extracting the root and dividing,
$$\frac{(a^2 + a_1^2 + a_2^2 + \&c.)^{\frac{1}{2}}}{(b^2 + b_1^2 + b_2^2 + \&c.)^{\frac{1}{2}}} = r = \frac{a}{b} = \frac{a_1}{b_1} = \&c.$$

Again, $na = rnb, \quad n_1a_1 = rn_1b_1, \quad n_2a_2 = rn_2b_2$, &c.
By addition,
$$(na + n_1a_1 + n_2a_2 + \&c.) = r\,(nb + n_1b_1 + n_2b_2 + \&c.)$$
Whence
$$\frac{na + n_1a_1 + n_2a_2 + \&c.}{nb + n_1b_1 + n_2b_2 + \&c.} = r = \frac{a}{b} = \frac{a_1}{b_1} = \&c.$$

II. If we wish to determine the variables from three simultaneous equations of the form
$$ax + by + cz = d \ldots\ldots (1),$$
$$a_1x + b_1y + c_1z = d_1 \ldots\ldots (2),$$
$$a_2x + b_2y + c_2z = d_2 \ldots\ldots (3),$$
instead of eliminating first z and then y, in order to determine x, we may eliminate both at one operation by the following rule: Multiply (1) by $b_1c_2 - c_1b_2$; (2) by $cb_2 - bc_2$; (3) by $bc_1 - b_1c$, and add: it will be found that the coefficients of y and z are identi-

cally equal to zero, and we have

$$x = \frac{d(b_1c_2 - c_1b_2) + d_1(cb_2 - bc_2) + d_2(bc_1 - b_1c)}{a(b_1c_2 - c_1b_2) + a_1(cb_2 - bc_2) + a_2(bc_1 - b_1c)};$$

with similar expressions for the other variables. If $d = 0$, $d_1 = 0$, $d_2 = 0$, the equations contain only two independent variables, since we may divide all by any one of them; and the condition that the equations should coexist is

$$a(b_1c_2 - c_1b_2) + a_1(cb_2 - bc_2) + a_2(bc_1 - b_1c) = 0.$$

We shall frequently refer to this process under the name of *cross-multiplication;* and the student is recommended to make himself familiar with the forms of the multipliers, as a ready use of the process will be of great service to him.

III. The sum of three squares of the form

$$(ay - bx)^2 + (cx - az)^2 + (bz - cy)^2$$

may be put in a shape which is very convenient, especially in geometrical investigations. For if we add and subtract from the preceding expression the three squares

$$a^2x^2, \quad b^2y^2, \quad c^2z^2,$$

the expression may be transformed into

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2,$$

or $$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\left\{1 - \frac{(ax + by + cz)^2}{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)}\right\}.$$

Now, if a, b, c be taken as proportional to the direction-cosines of some one line, and x, y, z of another, the expression

$$\frac{ax + by + cz}{(a^2 + b^2 + c^2)^{\frac{1}{2}}(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

is equal to the cosine of the angle between the lines: let this be θ; then the sum of the squares is equal to

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)(\sin\theta)^2.$$

IV. If we obtain as the result of any process that a function of x is equal to a function of y in which y is involved in a manner similar to that in which x is involved in the other, then, as there is nothing to distinguish one co-ordinate from another

when they are symmetrically involved, we may say that each of these functions is equal to a similar function of z, and this is the consequence of the general symmetry of our expressions. Thus, if we have two equations

$$lx + my + nz = 0,$$
$$l'x + m'y + n'z = 0,$$

and eliminate z between them, we find

$$(ln' - l'n)\,x + (mn' - m'n)\,y = 0,$$

or

$$\frac{x}{mn' - m'n} = \frac{y}{l'n - ln'};$$

here the two sides of the equation are symmetrical, one with respect to x and the other to y. We may therefore say that each is equal to $\dfrac{z}{lm' - l'm}$, this being the corresponding symmetrical expression with respect to z.

CHAPTER II.

OF THE STRAIGHT LINE AND PLANE.

(23) There are two methods which we may use in applying analysis to Geometry: either we may assume equations of different forms, and then determine their geometrical meaning; or we may define lines and surfaces by their geometrical properties, and from the definitions determine their equations. We might pursue either of these methods exclusively, and so build up a system on an uniform plan; but we shall find it more convenient to use sometimes one and sometimes the other method. In treating of the straight line and plane we shall use the second system, because their geometrical definitions are so well known, and their chief geometrical properties are so familiar to us, that it seems more natural to translate these into analytical language than to adopt the inverse process. The surfaces of the second order will be treated by the other method.

In the following investigations the co-ordinates are considered as rectangular, except when the contrary is expressly stated.

The Straight Line.

(24) *To find the equations to a straight line.* Take A a fixed point in the indefinite straight line AP (fig. 11), and let a, β, γ be its co-ordinates. Let x, y, z be the co-ordinates of any other point P in the line, and let l, m, n be the cosines of the angles which the line makes with the three axes, or the *direction-cosines* of the line. Then if r be the length of the portion AP of the line, $x - a$, $y - \beta$, $z - \gamma$ are the projections of r on the axes of x, y, z respectively. But by the nature of projections

$$x - a = lr, \quad y - \beta = mr, \quad z - \gamma = nr.$$

Hence we have $$\frac{x-a}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ \ldots\ldots\ldots\ldots(1).$$

These three equations are equivalent to two only, as any one may be derived from the other two; and as they express two relations between the co-ordinates of P which is any point in the straight line, they are the *equations to the line.*

As these formulæ express the equality of three ratios, it is often convenient to denote each of them by the single quantity r, to which they are each equal.

(25) It is evident that we may write the equations to the straight line under the form

$$\frac{x-a}{L}=\frac{y-\beta}{M}=\frac{z-\gamma}{N}\ \ldots\ldots\ldots\ldots(2),$$

if $L=kl$, $M=km$, $N=kn$, k being an arbitrary multiplier; that is, if L, M, N be proportional to l, m, n. Consequently if we have given equations of the form (2), they represent a straight line, and the quantities L, M, N are proportional to the direction-cosines of the line. The quantities a, β, γ are always the co-ordinates of a point through which the line passes.

(26) There is also another form in which the equations to the straight line may be written: for if we combine the first and third members of (2) and also the second and third, we have

$$x=a+\frac{L}{N}(z-\gamma),\quad y=\beta+\frac{M}{N}(z-\gamma),$$

which may be put in the form

$$x=az+p,\quad y=bz+q\ \ldots\ldots\ldots\ldots(3),$$

if $$\frac{L}{N}=a,\quad a-\frac{L}{N}\gamma=p,$$

$$\frac{M}{N}=b,\quad \beta-\frac{M}{N}\gamma=q.$$

The form (3) is that which has been usually employed by writers on this subject, but it is not so convenient as (1) and (2), because the expressions are not symmetrical with respect to the three variables x, y, z.

We can easily interpret the meanings of the constants in equations (3); for, considering the left-hand equation, we see

that it is the equation to a straight line in the plane of xz, which is evidently the projection of the given line on that plane. Now a is the tangent of the angle at which this projection cuts the axis of z, and p is the portion of the axis of x intercepted between the origin and the projection. In like manner b is the tangent of the angle which the projection of the given line on the plane of yz makes with the axis of z, and q is the portion of the axis of y which is intercepted between the origin and the projection.

(27) The position of the line will vary according to the values of the constants in its equations, and some of the more important cases we shall here consider.

In equations (2), if $\alpha = 0$, $\beta = 0$, $\gamma = 0$, the line passes through the origin, and its equations are then

$$\frac{x}{L} = \frac{y}{M} = \frac{z}{N} \dots\dots\dots\dots\dots (4).$$

If any one of the denominators vanish, as if $L = 0$, then, in order that the three equations may exist, the ratio $\frac{x - \alpha}{L}$ must not become infinite; and this can only be avoided by making it indeterminate, or of the form $\frac{0}{0}$, that is, by putting $x - \alpha = 0$. Now $L = 0$ implies that $l = 0$, or that the direction-cosine with respect to x is 0; that is, that the line is at right angles to the axis of x, and therefore parallel to the plane of yz. Hence, equations of the form

$$x = \alpha, \quad \frac{y - \beta}{M} = \frac{z - \gamma}{N} \dots\dots\dots\dots (5),$$

represent a straight line parallel to the plane of yz; and similarly for the other co-ordinate planes.

If two of the denominators vanish, as $L = 0$, $M = 0$, it follows that

$$x - \alpha = 0, \quad y - \beta = 0 \dots\dots\dots\dots (6),$$

which in this case are the equations to the line. Since $L = 0$, $M = 0$ imply that the line is perpendicular both to the axis of x and that of y, it must be parallel to the axis of z. A line, therefore, which is parallel to one axis is represented by making

the co-ordinates with respect to the other axes each equal to a constant.

If we consider the form (3) of the equations to the straight line, $p = 0$, $q = 0$ imply that the line passes through the origin, in which case its equations are

$$x = az, \quad y = bz.$$

If $a = 0$, the equations are

$$x = p, \quad y = bz + q,$$

which being of the same form as (5) represent a straight line parallel to the plane of yz. If $a = \infty$, which corresponds to $N = 0$ in the form (2), the equations (3) fail, and we must combine equations (2) in a different way. Let us then combine the first and second and the second and third, when we have

$$y = \beta + \frac{M}{L}(x - a), \quad z = \gamma + \frac{N}{L}(x - a);$$

in these, if $N = 0$, they become

$$y = a_1 x + p_1, \quad z = \gamma,$$

if
$$a_1 = \frac{M}{L} \text{ and } p_1 = \beta - \frac{M}{L} a.$$

These equations, by what has just been said, represent a straight line parallel to the plane of xy. Hence, if in equations (3) either of the coefficients of z becomes infinite, the equations represent a line parallel to the plane of xy.

We now proceed to apply these equations to the solutions of problems relating to straight lines.

(28) *To find the angles which a given straight line makes with the co-ordinate axes.*

Let the equations to the line be

$$\frac{x - a}{L} = \frac{y - \beta}{M} = \frac{z - \gamma}{N};$$

then if λ, μ, ν be the angles which the line makes with the axes, we have by Art. (25)

$$L = k \cos \lambda, \quad M = k \cos \mu, \quad N = k \cos \nu.$$

Squaring and adding these equations, we have, since

$$\cos^2 \lambda + \cos^2 \mu + \cos^2 \nu = 1,$$

$$L^2 + M^2 + N^2 = k^2,$$

and consequently

$$\cos\lambda=\frac{L}{\sqrt{(L^2+M^2+N^2)}},\ \cos\mu=\frac{M}{\sqrt{(L^2+M^2+N^2)}},\ \cos\nu=\frac{N}{\sqrt{(L^2+M^2+N^2)}}.$$

It is to be remarked that each of these expressions admits of two values in consequence of the double sign of the radical in the denominators, but as the same sign must be taken in each case, there are only two sets of values for the cosines, corresponding to the supplementary values of the angles λ, μ, ν, made with the positive axes of x, y, z by the two portions of the line measured in opposite directions from any point in it. It is necessary to make some convention respecting the mode in which the angles λ, μ, ν are to be measured, and that which is always used is that they are the angles made with the axes by that portion of the line which makes an acute angle with the positive axis of z. This implies that $\cos\nu$ is positive, and therefore that the radical has the same sign as N.

(29) *To find the equation to a straight line which passes through two given points.*

Let the co-ordinates of the points be $x_1, y_1, z_1, x_2, y_2, z_2$, and the assumed equations to the line be

$$\frac{x-a}{L}=\frac{y-\beta}{M}=\frac{z-\gamma}{N},$$

in which a, β, γ, L, M, N are to be determined in terms of $x_1, y_1, z_1, x_2, y_2, z_2$. In order that this line may pass through one of the given points, as the first, it is sufficient to make $a=x_1$, $\beta=y_1$, $\gamma=z_1$, as a, β, γ are the co-ordinates of some point through which the line passes. The equations then become

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N}.$$

But if the line is to pass through the point (x_2, y_2, z_2) these quantities must satisfy the preceding equations; therefore

$$\frac{x_2-x_1}{L}=\frac{y_2-y_1}{M}=\frac{z_2-z_1}{N}.$$

Eliminating L, M, N by dividing the first set of equations by

the second, member by member, we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

as the required equations.

(30) *To find the condition that two straight lines may intersect, and the position of the point of intersection.*

Since two straight lines in space are not necessarily in the same plane, and since two lines which intersect must be in the same plane, some relation must exist between the constants in the equations to the lines, in order that they may intersect, and the condition must be also that which holds in order that the two lines may be in one plane. Let the equations to the lines be

$$\frac{x - a}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \text{ .... } (1),$$

$$\frac{x - a'}{l'} = \frac{y - \beta'}{m'} = \frac{z - \gamma'}{n'} = r' \text{ .... } (2),$$

l, m, n, l', m', n' being the direction-cosines of the lines. These may be written in the forms

$$x = a + lr, \quad y = \beta + mr, \quad z = \gamma + nr \text{ ......} (1'),$$

$$x = a' + l'r', \quad y = \beta' + m'r', \quad z = \gamma' + n'r' \text{ ......} (2').$$

If the lines meet, the co-ordinates x, y, z of the point of intersection must satisfy the equations to both lines. Hence, x, y, z are the same in (1′) and (2′); therefore subtracting each equation of (2′) from the corresponding equation of (1′), we have

$$\left.\begin{array}{r} a - a' + lr - l'r' = 0, \\ \beta - \beta' + mr - m'r' = 0, \\ \gamma - \gamma' + nr - n'r' = 0. \end{array}\right\} \text{ .... } (3).$$

These three equations contain only two variables, r and r', and therefore, in order that they may co-exist, there must be some relation between the constants, which is the condition we are seeking. On eliminating r and r' by cross-multiplication (Art. 22), the condition is found to be

$$(m'n - mn')(a - a') + (ln' - l'n)(\beta - \beta') + (l'm - lm')(\gamma - \gamma') = 0 \text{...} (4).$$

To determine the position of the point of intersection, eliminate r between each pair of equations (3), which gives

$$(l'm - lm')r' = m(\alpha - \alpha') - l(\beta - \beta'),$$
$$(ln' - l'n)r' = l(\gamma - \gamma') - n(\alpha - \alpha'),$$
$$(m'n - mn')r' = n(\beta - \beta') - m(\gamma - \gamma').$$

Squaring and adding these, we have

$$r'^2 = \frac{(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2 - \{l(\alpha - \alpha') + m(\beta - \beta') + n(\gamma - \gamma')\}^2}{(l'm - lm')^2 + (ln' - l'n)^2 + (m'n - mn')^2},$$

so that r', the distance of the point of intersection from the point $(\alpha', \beta', \gamma')$, is determined.

If $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$, the equation of condition (4) is satisfied independently of $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, but then the value of r' becomes infinite, showing that the lines intersect at an infinite distance or are parallel, as will be seen in Art. (32).

If the equations to the straight lines be given under the forms

$$x = az + p, \quad y = bz + q,$$
$$x = a'z + p', \quad y = b'z + q',$$

the condition that they may intersect is

$$\frac{a - a'}{p - p'} = \frac{b - b'}{q - q'} \qquad \text{(5)},$$

(31) *To find the angle between two straight lines the equations to which are*

$$\frac{x - \alpha}{L} = \frac{y - \beta}{M} = \frac{z - \gamma}{N},$$
$$\frac{x - \alpha'}{L'} = \frac{y - \beta'}{M'} = \frac{z - \gamma'}{N'}.$$

By (18), if θ be the angle between the lines, and $\lambda, \mu, \nu, \lambda', \mu', \nu'$ be the angles which they make with the axes,

$$\cos\theta = \cos\lambda\cos\lambda' + \cos\mu\cos\mu' + \cos\nu\cos\nu'.$$

Substituting for the cosines their values given by (28), we have

$$\cos\theta = \frac{LL' + MM' + NN'}{(L^2 + M^2 + N^2)^{\frac{1}{2}}\,(L'^2 + M'^2 + N'^2)^{\frac{1}{2}}};$$

from which we also obtain

$$\sin\theta = \frac{\{(L'M - LM')^2 + (LN' - L'N)^2 + (M'N - MN')^2\}^{\frac{1}{2}}}{(L^2 + M^2 + N^2)^{\frac{1}{2}}(L'^2 + M'^2 + N'^2)^{\frac{1}{2}}}.$$

The expression for the cosine admits of two values, positive and negative, corresponding to the acute and the obtuse angles which the two lines make with each other. The value of the sine must always be taken as positive, since the angle between the lines can never exceed two right angles.

(32) *To find the conditions that two straight lines may be parallel or perpendicular to each other.*

If the two lines be parallel, their direction-cosines must be equal; and as L, M, N are proportional to the direction-cosines of the one line, and L', M', N' to those of the other line, these quantities must be proportional; or

$$\frac{L}{L'} = \frac{M}{M'} = \frac{N}{N'}$$

are the conditions of parallelism.

If the lines be perpendicular to each other, the cosine of the angle between them must be equal to 0, which, by the last article, gives

$$LL' + MM' + NN' = 0$$

as the condition of perpendicularity.

If instead of L, M, N, &c. we use the direction-cosines l, m, n', l', m', n, we may put the condition in either of the forms

$$ll' + mm' + nn' = 0,$$

or

$$(l'm - lm')^2 + (ln' - l'n)^2 + (m'n - mn')^2 = 1,$$

the latter being derived from the expression for $\sin\theta$ in Art. (31).

The Plane.

(33) *To find the equation to a plane.* For the purpose of investigating the equation to a plane, it may be defined as the surface traced out by a straight line which moves in such a manner as always to pass through one given straight line and to remain parallel to another. The moveable straight line is called the *generator*, and the fixed straight line through which it

always passes is called the *director*.* Let the equations to the director be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \;..........\; \text{(D)},$$

and those to the generator

$$\frac{x-x'}{l'}=\frac{y-y'}{m'}=\frac{z-z'}{n'}=r' \;........\; \text{(G)}.$$

Since the line (G) remains always parallel to a fixed straight line, while it passes through (D), its direction-cosines l', m', n' remain constant, while the values of x', y', z' vary in such a way as to satisfy the equations (D): this we may denote by putting x', y', z' for x, y, z in those equations. We then get

$$x'-\alpha=lr,\quad y'-\beta=mr,\quad z'-\gamma=nr,$$
$$x-x'=l'r',\quad y-y'=m'r',\quad z-z'=n'r'.$$

Adding these equations, we have

$$x-\alpha=lr+l'r' \;..............\; (1),$$
$$y-\beta=mr+m'r' \;............\; (2),$$
$$z-\gamma=nr+n'r' \;............\; (3).$$

In these equations r depends on the particular point in (D) through which (G) passes, and r' on the point in (G) which is under consideration; but we wish to find a relation between x, y, z, which shall be true for all points of (G) in every position. Such a relation it is plain we shall obtain by eliminating r and r' between equations (1), (2) and (3), since the result being independent of r, r' (the only quantities which particularize the position) must be true for all positions. The elimination is readily effected by cross-multiplication, which gives us

$$(m'n-mn')(x-\alpha)+(ln'-l'n)(y-\beta)+(l'm-lm')(z-\gamma)=0 \ldots (4).$$

This is a relation subsisting between the co-ordinates of every point of (G) in every position; in other words it is the equation to the plane which has been defined as the locus of (G).

* It has been usual, following the French fashion, to give these words a feminine termination, and to call them "generatrix" and "directrix;" but as it is not the custom of the English language to acknowledge the distinctions of gender in inanimate objects, I have taken the liberty of so far deviating from ordinary practice as to use that form which admits of a plural termination in our language; such words as *Generatrixes* and *Directrixes* being scarcely admissible.

(34) We may also adopt a more convenient method for the elimination of r and r' founded on geometrical considerations. Let $\cos\lambda$, $\cos\mu$, $\cos\nu$, be the direction-cosines of a line perpendicular to the plane containing, and consequently perpendicular to, both these lines, so that by (32) they satisfy the conditions

$$l\cos\lambda + m\cos\mu + n\cos\nu = 0,$$
$$l'\cos\lambda + m'\cos\mu + n'\cos\nu = 0.$$

Then if we multiply (1) by $\cos\lambda$, (2) by $\cos\mu$, (3) by $\cos\nu$, and add, the second side of the equation disappears in consequence of the preceding conditions, and there remains

$$(x-a)\cos\lambda + (y-\beta)\cos\mu + (z-\gamma)\cos\nu = 0 \;\ldots\ldots\; (5),$$

as the equation to the plane. It may also be written in the form

$$x\cos\lambda + y\cos\mu + z\cos\nu = a\cos\lambda + \beta\cos\mu + \gamma\cos\nu.$$

Now $\cos\lambda$, $\cos\mu$, $\cos\nu$, are the direction-cosines of a line perpendicular to the plane, and a, β, γ are the co-ordinates of some fixed point in the plane. Hence the second side of the equation is the sum of the projections of these co-ordinates on a line perpendicular to the plane. But this sum is equal to the perpendicular from the origin on the plane, since one extremity of the broken line $a+\beta+\gamma$, being the origin, coincides with one extremity of the perpendicular, while the other extremity is projected on the perpendicular by a line lying in the plane. Hence, calling the perpendicular from the origin δ, we have

$$a\cos\lambda + \beta\cos\mu + \gamma\cos\nu = \delta,$$

and $$x\cos\lambda + y\cos\mu + z\cos\nu = \delta, \;\ldots\ldots\ldots\; (6),$$

which is one of the most convenient forms of the equation to the plane.

As we shall have frequent occasion to speak of the line which is perpendicular to a plane, it will be convenient to have a distinct name for it, and we shall call it the *normal* to the plane, while we shall also call the direction-cosines of the line, or $\cos\lambda$, $\cos\mu$, $\cos\nu$, the direction-cosines of the plane, since they determine the position of the plane as much as that of the line.

(35) It appears then that the equation to the plane is of the first degree in x, y, z, and conversely we may show that the

general equation of the first degree can represent nothing but a plane.

The geometrical idea of a plane is that it is such a surface that a straight line which passes through any two points in it lies wholly in the surface. Now let

$$Ax + By + Cz = D \ \ldots\ldots\ldots\ldots \ (1)$$

be the general equation of the first degree, which by Art. (7) must represent some surface or other; and let $x_1, y_1, z_1, x_2, y_2, z_2$, be the co-ordinates of two points in the surface, and which therefore satisfy the preceding equation. Then the equations to a straight line passing through these two points are, by Art. (29),

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = r \ \ldots\ldots\ldots \ (2),$$

r being the length of the line between the point (x, y, z) and (x_1, y_1, z_1). Substituting in (1) for x, y, z, their values from (2), we have

$$\{A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1)\} r + Ax_1 + By_1 + Cz_1 = D \ldots (3).$$

But since $(x_1, y_1, z_1)(x_2, y_2, z_2)$ satisfy the equation (1), we have

$$Ax_1 + By_1 + Cz_1 = D = Ax_2 + By_2 + Cz_2;$$

hence equation (3) is satisfied identically, and therefore, whatever be the value of r and therefore of x, y, z, these quantities satisfy the equation (1). But (x, y, z) are the co-ordinates of any point in the line (2), consequently every point in the line lies in the surface represented by (1), which is therefore a plane.

(36) If the equation to the plane be given under the general form $$Ax + By + Cz = D,$$
it is easy to determine the geometrical meaning of the constants by comparing it with

$$x \cos \lambda + y \cos \mu + z \cos \nu = \delta.$$

This comparison gives

$$A = k \cos \lambda, \quad B = k \cos \mu, \quad C = k \cos \nu, \quad D = k\delta,$$

or
$$\frac{A}{\cos \lambda} = \frac{B}{\cos \mu} = \frac{C}{\cos \nu} = \frac{D}{\delta}.$$

That is, the coefficients of the variables are proportional to the

direction-cosines of the normal, and the constant term is proportional to the perpendicular from the origin on the plane.

(37) Let us now discuss the equation to the plane for different values of the constants.

If $D = 0$, the equation is

$$Ax + By + Cz = 0,$$

which is satisfied by $x = 0$, $y = 0$, $z = 0$, or the plane passes through the origin.

If $A = 0$, the equation is

$$By + Cz = D.$$

Now if $A = 0$, $\cos \lambda = 0$, or the normal is perpendicular to the axis of x, and therefore the plane itself is perpendicular to the plane of yz. In like manner, if either of the other coefficients of the variables vanish, we shall have an equation to a plane perpendicular to the co-ordinate plane containing the variables remaining in the equation. From this it is easy to see that the second form of the equations to the straight line given in (26) is equivalent to assigning the equations to two planes perpendicular to two of the co-ordinate planes, and so determining the position of the line.

If $B = 0$, $C = 0$, the equation is reduced to

$$Ax = D \text{ or } x = a \text{ a constant.}$$

Since $B = 0$ and $C = 0$, $\cos \mu = 0$, $\cos \nu = 0$, or the normal is perpendicular to the axis of y and that of z, and therefore to the plane containing them: hence the plane is parallel to the plane of yz. In like manner for the other axes we see that $y = b$ represents a plane parallel to xz, and $z = c$ one parallel to xy.

From this it is easy to see that

$$x = 0, y = 0, z = 0,$$

are the equations to the co-ordinate planes of yz, xz, xy, respectively.

(38) If in the general equation of Art. (36) we make one of the variables vanish as z, we obtain the equation to the intersection of the plane by the plane of xy. This equation is

$$Ax + By = D,$$

showing that the intersection is a straight line.

If we make $y = 0$, $z = 0$, we have

$$Ax = D,$$

as the equation for determining the point at which the plane cuts the axis of x. Let p be the distance of this point from the origin, then

$$p = \frac{D}{A}.$$

In like manner if q, r be the corresponding quantities for the other axes,

$$q = \frac{D}{B}, \quad r = \frac{D}{C}.$$

Hence the equation to the plane may be put under the form

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1,$$

which is often very convenient in practice.

The quantities p, q, r are called the *intercepts* on the axes.

(39) *To find the angles which a given plane makes with the co-ordinate planes.*

These angles are the same as those which the normal makes with the axes. If then the equation to the plane be

$$Ax + By + Cz = D,$$

the equations to the normal are

$$\frac{x-\alpha}{A} = \frac{y-\beta}{B} = \frac{z-\gamma}{C},$$

since A, B, C are proportional to the direction-cosines of the normal. If λ, μ, ν be the angles which this line makes with the axes of x, y, z, we have

$$\cos\lambda = \frac{A}{\sqrt{(A^2+B^2+C^2)}}, \quad \cos\mu = \frac{B}{\sqrt{(A^2+B^2+C^2)}}, \quad \cos\nu = \frac{C}{\sqrt{(A^2+B^2+C^2)}},$$

which expressions, therefore, determine the angles which the plane makes with the co-ordinate planes.

(40) *To find the angle between the straight line and plane, the equations to which are*

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n};$$

$$Ax + By + Cz = D.$$

By the angle between a line and a plane is meant the least angle which the line makes with any line in the plane; that is, the angle between the given line and its orthogonal projection on the plane. Hence, the given line, its projection, and the normal to the plane, lie all in one plane, and the angle between the line and its projection is the complement of the angle between the straight line and the normal. If this angle be ϕ, we have, by Art. (31),

$$\cos\phi = \frac{Al + Bm + Cn}{(A^2+B^2+C^2)^{\frac{1}{2}}},$$

so that if θ be the required angle,

$$\sin\theta = \frac{Al + Bm + Cn}{(A^2+B^2+C^2)^{\frac{1}{2}}}.$$

(41) *To find the angle between two planes, of which the equations are*

$$Ax + By + Cz = D,$$
$$A'x + B'y + C'z = D'.$$

The angle between two planes is the same as the angle between two lines drawn perpendicular to them; that is, it is equal to the angle between their normals. But the direction-cosines of the normals being proportional to A, B, C, A', B', C', we have, if θ be the angle between them,

$$\cos\theta = \frac{AA' + BB' + CC'}{(A^2 + B^2 + C^2)^{\frac{1}{2}}(A'^2 + B'^2 + C'^2)^{\frac{1}{2}}},$$

which expression therefore determines the angle between the two planes.

(42) *To find the conditions that two planes may be parallel or perpendicular.*

If the planes be parallel, their normals must also be parallel, and therefore, using the equations of the last article, we have, by Art. (32),

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'},$$

as the conditions of parallelism.

The condition of perpendicularity is at once obtained from the value of $\cos\theta$, for if the planes be perpendicular to each other, θ is a right angle, $\cos\theta = 0$, and therefore

$$AA' + BB' + CC' = 0.$$

(43) *To find the equation to a plane which passes through three given points.*

Since the equation to the plane contains only three independent constants, the three conditions of making the plane pass through the three given points are sufficient for determining the constants in the equation; or, in geometrical language, the position of a plane is determined by making it pass through three given points.

Let the co-ordinates of the three points be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and assume the equation of the plane to be

$$Ax + By + Cz = D;$$

it is required to find A, B, C, D, or rather the ratios of any three of them to the fourth in terms of the nine co-ordinates. Now if the plane passes through the point (x_1, y_1, z_1) those quantities must satisfy the equation to the plane, so that we have the condition

$$Ax_1 + By_1 + Cz_1 = D;$$

similarly

$$Ax_2 + By_2 + Cz_2 = D,$$

and

$$Ax_3 + By_3 + Cz_3 = D,$$

from which $\frac{A}{D}$, $\frac{B}{D}$ and $\frac{C}{D}$ are to be determined.

Eliminate $\frac{B}{D}$ and $\frac{C}{D}$ between these equations by cross-multiplication, so as to find $\frac{A}{D}$, similarly for $\frac{B}{D}$ and $\frac{C}{D}$: the common denominator of the fractions is

$$D = x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_2y_1z_3 + x_3y_1z_2 - x_3y_2z_1 ,$$

and the numerators are

$$A = y_2z_3 - y_3z_2 + y_3z_1 - y_1z_3 + y_1z_2 - y_2z_1 ,$$
$$B = z_2x_3 - z_3x_2 + z_3x_1 - z_1x_3 + z_1x_2 - z_2x_1 ,$$
$$C = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 .$$

The results of Chap. I. Arts. (20) and (21) enable us to assign geometrical meanings to these expressions; for if V be the volume of the tetrahedron, of which the origin is the vertex, and the three given points are the other angular points, then

$$D = -6V.$$

Also if A_x, A_y, A_z be the projections on the co-ordinate planes of the triangle, of which the three given points are the angular points,

$$A = -2A_x, \quad B = -2A_y, \quad C = -2A_z;$$

hence the equation to the plane may be put under the form

$$A_x x + A_y y + A_z z = 3V.$$

Problems relating to the Straight Line and Plane.

(44) *To find the equations to the line of intersection of two planes.*

Let the equations to the planes be

$$lx + my + nz = \delta$$
$$l'x + m'y + n'z = \delta'.$$

When these planes intersect, the co-ordinates x, y, z of the points of intersection are the same for both, and therefore the two equations may be taken as simultaneous, and combined accordingly. Eliminating then, first y and then x, we have

$$(l'm - lm')\, x = (m'n - mn')\, z + m\delta' - m'\delta,$$
$$(l'm - lm')\, y = (ln' - l'n)\, z + l'\delta - l\delta'.$$

These equations being of the form (3) of Art. (26), shew that the line of intersection is a straight line. If we wish to put the equations under the symmetrical form, let

$$m'n - mn' = \lambda, \quad ln' - l'n = \mu, \quad l'm - lm' = \nu.$$

Then eliminating z between the last two equations, we find

$$\frac{1}{\lambda}\left\{x + \frac{\lambda}{\mu\nu}(l'\delta - l\delta')\right\} = \frac{1}{\mu}\left\{y + \frac{\mu}{\lambda\nu}(m'\delta - m\delta')\right\}$$
$$= \frac{1}{\nu}\left\{z + \frac{\nu}{\lambda\mu}(n'\delta - n\delta')\right\},$$

by the symmetry of the formulæ.

If both planes pass through the origin, $\delta = 0$, $\delta' = 0$, and the equations to their line of intersection become simply

$$\frac{x}{mn' - m'n} = \frac{y}{l'n - ln'} = \frac{z}{lm' - l'm}.$$

(45) *To find the conditions that a straight line may be perpendicular to a plane.*

Let the equation to the plane be

$$Ax + By + Cz = D,$$

and the equations to the line

$$\frac{x-\alpha}{L} = \frac{y-\beta}{M} = \frac{z-\gamma}{N}.$$

Since the line is perpendicular to the plane it must be parallel to the normal of the plane; but the direction-cosines of the normal are proportional to A, B, C, and therefore, by Art. (32),

$$\frac{A}{L} = \frac{B}{M} = \frac{C}{N}$$

are the conditions of perpendicularity. Hence the equations to a line perpendicular to the plane and passing through (α, β, γ) are

$$\frac{x-\alpha}{A} = \frac{y-\beta}{B} = \frac{z-\gamma}{C}.$$

(46) *To find the condition that a straight line may be parallel to a plane.*

Take the same equations as in the last article; then, since the line is parallel to the plane, it must be perpendicular to the normal to the plane; wherefore, by Art. (32),

$$AL + BM + CN = 0$$

is the condition of parallelism.

(47) *To find the conditions that a given straight line may lie in a given plane.*

Since every point in the line lies in the plane, the co-ordinates α, β, γ must satisfy the equation to the plane, so that one condition is

$$A\alpha + B\beta + C\gamma = D.$$

But a line which lies in a plane must be perpendicular to the normal; this gives, as the other condition,

$$AL + BM + CN = 0.$$

(48) *To find the length of the perpendicular drawn from a given point on a plane, the equation to which is given.*

Let x', y', z' be the co-ordinates of the point, and

$$Ax + By + Cz = D \ldots\ldots\ldots\ldots\ldots (1),$$

the equation to the plane. The equations to a line perpendicular to the plane and passing through (x', y', z') are

$$\frac{x-x'}{A}=\frac{y-y'}{B}=\frac{z-z'}{C} \quad \ldots\ldots\ldots\ldots \quad (2).$$

Each of these ratios is, by the Theorem I. of Chap. I., equal to

$$\frac{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}}{(A^2+B^2+C^2)^{\frac{1}{2}}};$$

and also to

$$\frac{A(x-x')+B(y-y')+C(z-z')}{A^2+B^2+C^2}.$$

Now if (x, y, z) be the point where (2) meets (1), the numerator of the former of these is the perpendicular distance of the point from the plane. Let this be δ; then, as (x, y, z) is a point in the plane, we have, by (1),

$$Ax+By+Cz=D,$$

and consequently

$$\delta=\frac{D-(Ax'+By'+Cz')}{(A^2+B^2+C^2)^{\frac{1}{2}}}.$$

If the numerator be negative we must then take the denominator with the negative sign, since the value of δ is absolute, and must therefore be considered as positive.

(49) *To find the length of the perpendicular from a given point on a given straight line.*

Let the co-ordinates of the given point be x', y', z', and the equations to the given line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r.$$

Assume the equation to the line perpendicular to it and passing through (x', y', z') to be

$$\frac{x-x'}{l'}=\frac{y-y'}{m'}=\frac{z-z'}{n'}=r'.$$

Then if these lines intersect in the point (x, y, z), r' is the length of the perpendicular required. Now, eliminating x, y, z between the corresponding pairs of equations, we have

$$x'-\alpha=lr-l'r', \quad y'-\beta=mr-m'r', \quad z'-\gamma=nr-n'r'.$$

Squaring and adding, and observing that, since the lines are perpendicular to each other,

$$ll' + mm' + nn' = 0,$$

we have $\qquad (x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = r^2 + r'^2.$

Again, multiply by l, m, n, and add, then

$$l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma) = r\,;$$

consequently we have

$$r'^2 = (x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 - \{l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma)\}^2,$$

which determines r', the length of the perpendicular.

(50) *To find the perpendicular distance between two straight lines not in the same plane.*

Let the equations to the lines be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \;\ldots\ldots\ldots\ldots (1),$$

$$\frac{x - \alpha'}{l'} = \frac{y - \beta'}{m'} = \frac{z - \gamma'}{n'} \;\ldots\ldots\ldots\ldots (2).$$

Through (1) draw a plane parallel to (2), and another through (2) parallel to (1). These planes will then be parallel to each other and at right angles to the perpendicular distance between the two lines. Their equations, therefore, will be of the forms

$$(x - \alpha)\lambda + (y - \beta)\mu + (z - \gamma)\nu = 0 \;\ldots\ldots\ldots (3),$$

$$(x - \alpha')\lambda + (y - \beta')\mu + (z - \gamma')\nu = 0 \;\ldots\ldots\ldots (4),$$

the constants λ, μ, ν being determined by the equations of condition,

$$l\lambda + m\mu + n\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$l'\lambda + m'\mu + n'\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\lambda^2 + \mu^2 + \nu^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

the last equation implying that we assume the constants to be the direction-cosines of the plane. To determine their actual values we proceed as follows: eliminate ν between (5) and (6), when we obtain

$$(ln' - l'n)\lambda + (mn' - m'n)\mu = 0\,;$$

whence $\qquad \dfrac{\lambda}{mn' - m'n} = \dfrac{\mu}{l'n - ln'} = \dfrac{\nu}{lm' - l'm}$

by the symmetry of the formulæ.

Assume each of these expressions equal to u, and substitute the values of λ, μ, ν in (7), we then obtain

$$u^2 \{(mn' - m'n)^2 + (l'n - ln')^2 + (lm' - l'm)^2\} = 1.$$

Now, by Art. (22), the multiplier of u^2 is equal to the square of the sine of the angle contained by the lines whose direction-cosines are l, m, n, l', m', n', that is, by the given lines. Let this angle be θ, then we have $u \sin\theta = \pm 1$, and therefore

$$\lambda = \pm \frac{mn' - m'n}{\sin\theta}, \quad \mu = \pm \frac{l'n - ln'}{\sin\theta}, \quad \nu = \pm \frac{lm' - l'm}{\sin\theta}.$$

These constants being thus determined, it is easy to find the perpendicular distance required, for it is evidently equal to the difference of the perpendiculars from the origin on the planes (3) and (4). That difference is

$$(\alpha \sim \alpha')\lambda + (\beta \sim \beta')\mu + (\gamma \sim \gamma')\nu,$$

so that if δ be the required distance

$$\delta = \frac{(mn' - m'n)(\alpha \sim \alpha') + (l'n - ln')(\beta \sim \beta') + (lm' - l'm)(\gamma \sim \gamma')}{\sin\theta}.$$

(51) *To find the shortest distance between two straight lines.*

Let the equations to the lines be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r,$$

$$\frac{x'-\alpha'}{l'} = \frac{y'-\beta'}{m'} = \frac{z'-\gamma'}{n'} = r'.$$

If δ be the distance between any two points in the lines

$$\delta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$

which is to be a minimum. Now x, y, z are functions of r, and x', y', z' of r', and r and r' are independent; therefore, making the differential of δ equal to 0, we have two equations

$$(x-x')\,dx + (y-y')\,dy + (z-z')\,dz = 0,$$

$$(x-x')\,dx' + (y-y')\,dy' + (z-z')\,dz' = 0.$$

But from the equations to the lines

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n} \quad \text{and} \quad \frac{dx'}{l'} = \frac{dy'}{m'} = \frac{dz'}{n'};$$

hence, eliminating the differentials by dividing each term of the

former equations by the corresponding member of the latter, we have

$$l(x-x')+m(y-y')+n(z-z')=0,$$
$$l'(x-x')+m'(y-y')+n'(z-z')=0.$$

Now if λ, μ, ν be the direction-cosines of the line δ,

$$x-x'=\lambda\delta,\quad y-y'=\mu\delta,\quad z-z'=\nu\delta,$$

and the preceding equations become

$$l\lambda+m\mu+n\nu=0,$$
$$l'\lambda+m'\mu+n'\nu=0.$$

These conditions show that the least distance is perpendicular to both the given lines, and hence its length is given by the solution of the problem in the last article.

(52) *To find the equations to the straight line which cuts at right angles two given straight lines.*

Let the equations to the given lines be

$$\frac{x-a}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ \ldots\ldots\ldots\ldots(1),$$

$$\frac{x-a'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}\ \ldots\ldots\ldots\ldots(2);$$

and assume the required equations to be

$$\frac{x-x'}{\lambda}=\frac{y-y'}{\mu}=\frac{z-z'}{\nu}\ \ldots\ldots\ldots\ldots(3).$$

The values of λ, μ, ν are evidently the same as those found in Art. (50), and we have only to determine x', y', z'; now these being the co-ordinates of an arbitrary point in the line, we may assume that point to be the intersection of (1) and (3), so that these quantities must satisfy the equation (1), or

$$\frac{x'-a}{l}=\frac{y'-\beta}{m}=\frac{z'-\gamma}{n}=r,\ \text{suppose}\ \ldots\ldots(4).$$

The condition that (2) and (3) should intersect, is

$$(x'-a')(n'\mu-m'\nu)+(y'-\beta')(l'\nu-n'\lambda)+(z'-\gamma')(m'\lambda-l'\mu)=0;$$

and, as $x'-a'=x'-a-(a'-a)$, and similarly for the other quantities, this equation may be written

$$(x'-a)(n'\mu-m'\nu)+(y'-\beta)(l'\nu-n'\lambda)+(z'-\gamma)(m'\lambda-l'\mu)=$$
$$(a'-a)(n'\mu-m'\nu)+(\beta'-\beta)(l'\nu-n'\lambda)+(\gamma'-\gamma)(m'\lambda-l'\mu)\ \ldots\ (5).$$

Substitute in (5) for $(x'-\alpha)$, $(y'-\beta)$, $(z'-\gamma)$, their values derived from (4), so as to obtain an equation in r, which gives

$$r=\frac{(\alpha'-\alpha)(n'\mu-m'\nu)+(\beta'-\beta)(l'\nu-n'\lambda)+(\gamma'-\gamma)(m'\lambda-l'\mu)}{l(n'\mu-m'\nu)+m(l'\nu-n'\lambda)+n(m'\lambda-l'\mu)}.$$

The value of r being thus found, those of x', y', z' are known from equations (4), and thus the line is completely determined. If we substitute for λ, μ, ν their values, and reduce by means of the conditions

$$l^2+m^2+n^2=1,\quad ll'+mm'+nn'=\cos\theta,$$

where θ is the angle between the lines (1) and (2), we obtain

$$r=\frac{(\alpha'-\alpha)(l-l'\cos\theta)+(\beta'-\beta)(m-m'\cos\theta)+(\gamma'-\gamma)(n-n'\cos\theta)}{\sin^2\theta}.$$

Oblique Co-ordinates.

(53) The equations to a straight line when referred to oblique co-ordinates are in the same form as when the co-ordinates are rectangular; viz. $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n};$$

where however l, m, n no longer signify the direction-cosines of the lines, but the ratios of the projections on the axes of any portion of the line to that portion, the projections being made by planes parallel to the co-ordinate planes. For if r be any portion of the line between the points $(x, y, z)(\alpha, \beta, \gamma)$ its projections are

$$lr,\quad mr,\quad nr;$$

but these projections are also

$$x-\alpha,\quad y-\beta,\quad z-\gamma,$$

whence, equating these values, we obtain

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r.$$

The equations of a straight line passing through two given points $(x_1, y_1, z_1)(x_2, y_2, z_2)$ are the same in form as for rectangular co-ordinates; that is, they are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

(54) The quantities l, m, n are not independent, but are connected by an equation of condition which may be found as

follows. The distance of the point (x, y, z) from (α, β, γ) is, by Chap. I. Art. (15), given by the formula

$$r^2=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2+2a(y-\beta)(z-\gamma)+2b(x-\alpha)(z-\gamma)+2c(x-\alpha)(y-\beta),$$

where a, b, c are the cosines of the angles between the axes of yz, xz, xy. But from the equation to the line, we have

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,$$

and hence eliminating $x-\alpha$, $y-\beta$, $z-\gamma$, we find

$$1=l^2+m^2+n^2+2amn+2bln+2clm,$$

as the equation connecting l, m, n. When the co-ordinates are rectangular, $a=0$, $b=0$, $c=0$, and the condition is reduced to

$$1=l^2+m^2+n^2,$$

as it manifestly ought to be.

(55) *To find the angle between two lines, the equations to which are*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}; \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}.$$

Through the origin draw two lines parallel to the given lines, so that their equations are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r, \quad \frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}=r'.$$

Now if θ be the angle between the lines, δ the distance between the extremities of r and r',

$$\delta^2=r^2-2rr'\cos\theta+r'^2;$$

but $(x, y, z)(x', y', z')$ being the co-ordinates of the extremities of δ, we have also

$$\begin{aligned}\delta^2&=(x-x')^2+(y-y')^2+(z-z')^2+2a(y-y')(z-z')\\&\quad+2b(x-x')(z-z')+2c(x-x')(y-y'),\\&=r^2+r'^2-2\{xx'+yy'+zz'+a(yz'+y'z)+b(xz'+x'z)+c(xy'+x'y)\}.\end{aligned}$$

Equating these two values of δ^2, and eliminating x, y, z, x', y', z' by means of the equations to the lines, and then dividing by rr', we find

$$\cos\theta=ll'+mm'+nn'+a(mn'+m'n)+b(ln'+l'n)+c(lm'+l'm);$$

by which the angle θ is determined.

If λ, μ, ν be the angles which the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

makes with the axes, they may be determined by the formula just found. For let the second line coincide with the axis of x, then $l' = 1$, $m' = 0$, $n' = 0$, and $\theta = \lambda$, so that

$$\cos \lambda = l + bn + cm.$$

Similarly

$$\cos \mu = m + an + cl,$$
$$\cos \nu = n + am + bl.$$

If we multiply these equations by l', m', n' respectively, and add, we find

$$\cos \theta = l' \cos \lambda + m' \cos \mu + n' \cos \nu$$

identical in form with the equation in rectangular co-ordinates, though the quantities involved have not the same meanings.

The condition that the two lines may be perpendicular to each other is evidently

$$l' \cos \lambda + m' \cos \mu + n' \cos \nu = 0;$$

or $ll' + mm' + nn' + a(mn' + m'n) + b(ln' + l'n) + c(lm' + l'm) = 0$;
and the conditions that they may be parallel are

$$\frac{l + bn + cm}{l' + bn' + cm'} = \frac{m + an + cl}{m' + an' + cl'} = \frac{n + am + bl}{n' + am' + bl'}.$$

(56) The equation to the plane when referred to oblique co-ordinates must evidently be of the same form as for rectangular co-ordinates, viz.

$$Ax + By + Cz = D,$$

since the equations to the straight lines, from which it is derived, are of the same form in both cases.

If the equation be in the form

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1,$$

p, q, r are the intercepts of the axes, exactly as with rectangular co-ordinates.

(57) *To find the conditions that the straight line*

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \ldots\ldots\ldots\ldots (1)$$

may be perpendicular to the plane

$$Ax + By + Cz = D \ldots\ldots\ldots\ldots (2).$$

If the straight line be perpendicular to the plane, it must be perpendicular to every straight line in the plane. Let the equations to any one of them be

$$\frac{x-\alpha}{l'} = \frac{y-\beta}{m'} = \frac{z-\gamma}{n'} \ldots\ldots\ldots (3);$$

then l', m', n' must satisfy the equation

$$Al' + Bm' + Cn' = 0 \ldots\ldots\ldots\ldots (4),$$

since the line lies wholly in the plane. But the condition that the line (1) should be perpendicular to (3) is, by Art. (55),

$$ll' + mm' + nn' + a(mn' + m'n) + b(ln' + l'n) + c(lm' + l'm) = 0 \ldots (5).$$

The equation (5) is to subsist for all values of l', m', n' which satisfy the condition (4); therefore if we were to eliminate one of them as n', we should have an equation involving the other two l' and m', and as these are independent, their coefficients must separately vanish. But it is more convenient to use the method of indeterminate multipliers. Multiply then (4) by a quantity k, and subtract from it (5); then, if we assume as the condition for determining k that the coefficient of l' shall vanish, we have

$$kA = l + cm + bn;$$

and as m', n' are independent, their coefficients must vanish separately: hence

$$kB = cl + m + an,$$
$$kC = bl + am + n,$$

which are the three required conditions. The quantity k is easily determined; for if we multiply by l, m, n, and add, we find, by the condition of Art. (54),

$$k(Al + Bm + Cn) = 1, \quad \text{or} \quad k = \frac{1}{Al + Bm + Cn}.$$

(58) Oblique co-ordinates may be conveniently used for demonstrating various properties of the tetrahedron.

1st, The straight line joining the middle points of opposite edges all pass through one point. Take O (fig. 12) one of the summits as origin, and the three contiguous edges OA, OB, OC as the axes of x, y, z. Then if P be the middle point of AB

and Q of OC, PQ is one of the lines which we have to consider. The general equations to a line passing through two points are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

Let $OA = 2a$, $OB = 2b$, $OC = 2c$; then the co-ordinates of $Q(x_1, y_1, z_1)$ are 0, 0, c, and those of $P(x_2, y_2, z_2)$ are a, b, 0; so that the equations to PQ are

$$\frac{x}{a} = \frac{y}{b} = \frac{z-c}{-c}.$$

In like manner the equations to the line joining the middle points of OB and AC are

$$\frac{x}{a} = \frac{y-b}{-b} = \frac{z}{c},$$

and those of that joining the middle points of OA and BC are

$$\frac{x-a}{-a} = \frac{y}{b} = \frac{z}{c}.$$

Combining the first and second equations, we find

$$x = \frac{a}{2}, \quad y = \frac{b}{2}, \quad z = \frac{c}{2},$$

values which also satisfy the third equations; consequently all three lines pass through the point of which these are the co-ordinates.

2nd, If through three conterminous edges planes be drawn bisecting the opposite edges, they will intersect each other in the same straight line. The figure being the same as before, and the edges at O being taken as those through which the planes are to pass, the equation to the plane passing through OC and bisecting AB is

$$\frac{x}{a} - \frac{y}{b} = 0.$$

Similarly the other two equations are

$$\frac{z}{c} - \frac{x}{a} = 0,$$

$$\frac{y}{b} - \frac{z}{c} = 0,$$

and it is easily seen that these are satisfied by the relations

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c},$$

which are therefore the equations to one straight line in which the three planes intersect.

3rd, The six planes which pass through the six edges of the tetrahedron and bisect the opposite edges, all pass through one point.

Three of these planes are those which have been considered in the last problem, and it is obvious that their equations are satisfied by the co-ordinates of the point found in the first problem, viz.

$$x = \frac{a}{2}, \quad y = \frac{b}{2}, \quad z = \frac{c}{2}.$$

But that point is, by construction, symmetrical with respect to the tetrahedron, and therefore the three planes which pass through the edges terminated at A, will also pass through it, and hence that point is common to all the six planes.

The point in question is the centre of gravity of the solid.

CHAPTER III.

TRANSFORMATION OF CO-ORDINATES.

R

As the origin and direction of the axes to which the position of a point in space is referred are quite arbitrary, and as the simplicity of our expressions may be very much affected by the choice which we make of these, we proceed to establish formulæ for changing one system of co-ordinates to another.

(59) *To change the origin of co-ordinates, the axes remaining parallel to their original position.*

Let O (fig. 13) be the origin of the old axes Ox, Oy, Oz; O' the origin of the new axes $O'x'$, $O'y'$, $O'z'$, parallel to the former. Let the co-ordinates of O' referred to the old axes be

$$OQ = \alpha, \quad QR = \beta, \quad O'R = \gamma.$$

Let P be any point in space, and let its co-ordinates referred to the old axes be

$$OM = x, \quad MN = y, \quad PN = z,$$

and those referred to the new axes

$$O'M' = x', \quad M'N' = y', \quad PN' = z'.$$

Then, as
$$OM = OQ + QM = OQ + O'M',$$
$$MN = QR + LN = QR + M'N',$$
$$PN = NN' + PN' = O'R + PN',$$

we have
$$x = \alpha + x', \quad y = \beta + y', \quad z = \gamma + z' \ldots . (1),$$

as the expressions for the old co-ordinates in terms of the new. These being substituted in any function of the variables x, y, z, give a result involving x', y', z', and therefore referred to the new co-ordinates.

In the figure O' has been assumed to lie within the positive axes of the old system, and P within those of the new. But if

either chance to lie in the direction of any negative axis, the formula is easily adapted to such a case by a change of sign. Thus, if O'' be the new origin,

$$OM = MQ' - OQ' = M'O'' - OQ',$$

or

$$x = x' - a.$$

In like manner, if P lie towards the negative axis of y',

$$y = \beta - y';$$

and so forth. Hence, the formulæ (1) are true for all cases if we attach to the quantities involved their proper signs depending on their positions relative to the origins. These formulæ hold equally for rectangular and oblique co-ordinates.

(60) *To pass from a rectangular system to any other, the origin remaining the same.*

Let Ox, Oy, Oz (fig. 14) be the old axes at right angles to each other, Ox', Oy', Oz' the new axes inclined to each other at any angle,

$$OM = x, \quad MN = y, \quad NP = z,$$
$$OM' = x', \quad M'N' = y', \quad N'P = z'.$$

Project the broken line $x' + y' + z'$ on the axis Ox, by drawing from M', N', P perpendiculars to that line; the last one PM falls at the extremity of the abscissa x. Hence the line OM or x is equal to the sum of the projections of x', y', and z'. Let a, b, c be the cosines of the angles which the new axes make with the axis Ox; then, by the theory of projections,

$$x = ax' + by' + cz'.$$

We have here assumed that a, b, c are the cosines of the angles which the *positive new axes* make with the positive old axis of x, and that in the figure each of the axes makes an acute angle with Ox. But if, as in fig. (15), one of the new axes, as Oz', makes an obtuse angle with Ox, we shall have

$$OM = Om' + m'n' - Mn',$$

or

$$x = ax' + by' - cz',$$

which is equivalent to the previous formula, as c would in this case become negative, as it is the cosine of an angle greater than a right angle.

If one of the new co-ordinates, as x', fall on the other side of the origin, the term involving it would be subtracted from the others; and hence in this case also the original formula would apply by reckoning x' as negative. Therefore in all cases we have

$$\left.\begin{aligned} x &= ax' + by' + cz', \\ y &= a'x' + b'y' + c'z', \\ z &= a''x' + b''y' + c''z', \end{aligned}\right\} \ldots\ldots\ldots (2),$$

and similarly,

where a', b', c' are the cosines of the angles which the new axes make with the old axis of y, and a'', b'', c'' of those which they make with the old axis of z. These nine quantities are connected by certain conditions: for since Ox' is a line, of which the direction-cosines are a, a', a'', we have, by Art. (16),

$$\left.\begin{aligned} a^2 + a'^2 + a''^2 &= 1, \\ b^2 + b'^2 + b''^2 &= 1, \\ c^2 + c'^2 + c''^2 &= 1. \end{aligned}\right\} \ldots\ldots\ldots (3).$$

similarly, and

(61) *To pass from one system of rectangular co-ordinates to another also rectangular.*

The formulæ in this case are the same as those in the last article, with the addition of the conditions expressing the perpendicularity of the new axes. These conditions are evidently, by Art. (32),

$$\left.\begin{aligned} ab + a'b' + a''b'' &= 0, \\ ac + a'c' + a''c'' &= 0, \\ bc + b'c' + b''c'' &= 0. \end{aligned}\right\} \ldots\ldots\ldots (4).$$

Since between the nine quantities there are six equations of conditions, there are only three of the quantities a, b, c, &c. independent.

(62) There is another set of conditions equivalent to the preceding, which may be deduced from the consideration that the new axes being rectangular, any one of the old axes bears to the new system the same relation as the corresponding one of the new system bears to the old one, and therefore we must have

$$\left.\begin{aligned} a^2 + b^2 + c^2 &= 1 \\ a'^2 + b'^2 + c'^2 &= 1 \\ a''^2 + b''^2 + c''^2 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (5),$$

$$\left.\begin{aligned} aa' + bb' + cc' &= 0 \\ aa'' + bb'' + cc'' &= 0 \\ a'a'' + b'b'' + c'c'' &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (6).$$

These conditions are not different from the former, as the one may be deduced from the other by the relation

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2.$$

(63) We may also express x', y', z' in terms of x, y, z; for, multiply equations (2) by a, a', a'' respectively, and add; then, attending to the conditions (3) and (4),

$$\left.\begin{aligned} x' &= ax + a'y + a''z, \\ \text{similarly} \quad y' &= bx + b'y + b''z, \\ \text{and} \quad z' &= cx + c'y + c''z, \end{aligned}\right\} \ldots\ldots\ldots\ldots (7).$$

(64) Besides these there are some other very remarkable relations between the quantities a, b, c, &c., first given by Lagrange, *Mec. Anal.* vol. II. p. 217.

Eliminate y' and z' from equations (2) by cross-multiplication; we then obtain

$$kx' = (b'c'' - b''c')\,x + (b''c - bc'')\,y + (bc' - b'c)\,z,$$

where $\quad k = a\,(b'c'' - b''c') + a'\,(b''c - bc'') + a''\,(bc' - b'c).$

Comparing this with the first of equations (7), we have

$$a = \frac{b'c'' - b''c'}{k}, \quad a' = \frac{b''c - bc''}{k}, \quad a'' = \frac{bc' - b'c}{k}.$$

On substituting these values in the first of the conditions (3), we find

$$\begin{aligned} k^2 &= (b'c'' - b''c')^2 + (b''c - bc'')^2 + (bc' - b'c)^2 \\ &= (b^2 + b'^2 + b''^2)(c^2 + c'^2 + c''^2)\sin^2\theta, \end{aligned}$$

by Theorem III. Art. (22); where θ is the angle between the lines whose direction-cosines are b, b', b'', c, c', c''; but that angle is a right angle, and therefore $\sin\theta = 1$; hence $k^2 = 1$, and therefore $k = \pm 1$; hence

$$\left.\begin{aligned} &a = \pm(b'c'' - b''c'), \quad a' = \pm(b''c - bc''), \quad a'' = \pm(bc' - b'c) \\ &\text{similarly} \\ &b = \pm(a''c' - a'c''), \quad b' = \pm(ac'' - a''c), \quad b'' = \pm(a'c - ac') \\ &c = \pm(a'b'' - b'a''), \quad c' = \pm(a''b - ab''), \quad c'' = \pm(ab' - a'b) \end{aligned}\right\} \ldots (8).$$

(65) *Euler's formulæ for transforming from one system of rectangular co-ordinates to another.*

The preceding symmetrical transformation involves nine quantities connected by six equations of condition, so that there are only three independent quantities. Hence, it ought to be possible to effect the transformation by means of three quantities only. The three which Euler has chosen for the purpose are—the angle which the trace of the new plane of xy on the old plane of xy makes with the old axis of x, the angle which the trace makes with the new axis of x, and the inclination of the two planes of xy, that is, the angle between the old and new axis of z.

Let Ox, Oy, Oz (fig. 16) be the old axes, Ox', Oy', Oz' the new axes; Ox_1 the trace of the plane of $x'y'$ on that of xy; then the position of the new axes with respect to the old is known if we know the angles

$$xOx_1 = \phi, \quad x'Ox_1 = \psi, \quad zOz' = \theta.$$

The required formulæ may be most readily proved by successive transformations, each in one plane only. Thus, keeping the axis of z unaltered, let us turn the axes of x and y in their own plane till Ox coincides with Ox_1, so that the new rectangular system consists of Ox_1, Oz, and an axis Oy_1 perpendicular to them. Let x_1, y_1, z be the co-ordinates of a point referred to the new axes; then, since the axis of x_1 makes an angle ϕ with the axis of x,

$$x = x_1 \cos\phi - y_1 \sin\phi,$$
$$y = x_1 \sin\phi + y_1 \cos\phi.$$

Again, the axis Ox_1 remaining fixed, turn the axes Oz and Oy_1 through an angle θ in their own plane; then if Oz', Oy_2 be the new axes,

$$z = z' \cos\theta - y_2 \sin\theta,$$
$$y_1 = z' \sin\theta + y_2 \cos\theta.$$

Lastly, the axis Oz' remaining fixed, turn the axes Ox_1, Oy_2 in their own plane through an angle ψ, so that they take the positions Ox', Oy'; then

$$x_1 = x' \cos\psi - y' \sin\psi,$$
$$y_2 = x' \sin\psi + y' \cos\psi.$$

By means of the last three equations we can eliminate x_1, y_1, y_2 from the first three, and we find

$$\left.\begin{aligned}
x &= x'(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta)\\
&\quad - y'(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta)\\
&\quad - z'\sin\phi\sin\theta\\
y &= x'(\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta)\\
&\quad - y'(\sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta)\\
&\quad - z'\cos\phi\sin\theta\\
z &= z'\cos\theta - x'\sin\theta\sin\psi - y'\sin\theta\cos\phi
\end{aligned}\right\}\ldots\ldots(9),$$

which are the required formulæ of transformation. These expressions are essentially unsymmetrical, and they are so cumbrous that it is always desirable to avoid using them if other means for effecting the required transformation can be found. We shall never employ them, and they are introduced here only because a knowledge of them is assumed in some parts of Dynamics.

(66) *To pass from one system of oblique co-ordinates to another also oblique.*

This is best done by the aid of orthogonal projections; but instead of using separate letters to designate the cosines of the various angles, we shall represent the angle between two lines, such as the axes of x and of y, by the symbol $\widehat{x, y}$, and similarly for the others. Let the co-ordinates of a point in space be,

when referred to the old axes, x, y, z,

when referred to the new axes, x', y', z'.

At the origin O draw a normal to the plane of (x, y) in the direction of the positive axis of z, and designate the normal by the letter n. Project on this normal the broken line $x + y + z$, terminated by the origin and the point. The projections of x and y are evidently zero, and there remains

$$z\cos\widehat{n, z}.$$

Again, project the broken line $x' + y' + z'$ on the same line: this gives us

$$x'\cos\widehat{n, x'} + y'\cos\widehat{n, y'} + z'\cos\widehat{n, z'},$$

which must be equal to the preceding projection, as the extremities of the two broken lines are the same: hence

$$\left.\begin{array}{l} z\cos\widehat{n,z} = x'\cos\widehat{n,x'} + y'\cos\widehat{n,y'} + z'\cos\widehat{n,z'} \\ \text{In like manner we can find} \\ y\cos\widehat{n',y} = x'\cos\widehat{n',x'} + y'\cos\widehat{n',y'} + z'\cos\widehat{n',z'} \\ x\cos\widehat{n'',z} = x'\cos\widehat{n'',x'} + y'\cos\widehat{n'',y'} + z'\cos\widehat{n'',z'} \end{array}\right\}\ldots\ldots(10),$$

n' and n'' being normals to the planes of xz and yz respectively. These formulæ were first given by Français, *Jour. Polytechnique*, (cap. XIV.); although elegant they are of little practical utility.

(67) It is to be remarked that the transformation of co-ordinates can never affect the degree of the equation; that is, it can never increase or diminish the greatest sum of the indices of the variables. In the first place it cannot increase it, for if $Ax^m y^n z^p$ be any term in the equation, its order being $m + n + p$, and we have in changing co-ordinates to substitute for x, y, z, expressions of the form

$$ax' + by' + cz' + d,$$

so that the term becomes

$$A(ax' + by' + cz' + d)^m (a'x' + b'y' + c'z' + d')^n (a''x' + b''y' + c''z' + d'')^p.$$

The highest power in this has for its index $m + n + p$, the same as before; hence no term can, by changing co-ordinates, introduce a term of a higher degree, and consequently the degree of the equation cannot be raised.

Neither in the second place can it be lowered, for if so, it would be impossible, by what we have just shewn, to bring back the equation to its primitive form by any change of co-ordinates.

(68) It was shown in Art. (10) how we could determine the curve of intersection of a surface with any one of the co-ordinate planes, or with a plane parallel to one of them. But when the position of the plane is general, the determination is not so simple; for if we combine the equation to a surface

$$f(x, y, z) = 0,$$

with that to a plane which cuts it,

$$Ax + By + Cz = D,$$

so as to eliminate one of the variables as z, we obtain an equation

$$\phi(x, y) = 0,$$

which is, not the equation to the curve of intersection, but that to its projection on the plane of (x, y). In order, therefore, to determine absolutely the curve of intersection, we must change the direction of co-ordinates until one of the co-ordinate planes, as (x, y), is parallel to the cutting plane. For this purpose we must know the inclination of the cutting plane to one of the co-ordinate planes, and the angle which its trace on that plane makes with one of the axes in that plane.

Now if θ be the inclination of the cutting plane to the plane of (x, y), we have, by Art. (39),

$$\cos\theta = \frac{C}{(A^2 + B^2 + C^2)^{\frac{1}{2}}},$$

and if ϕ be the angle which the trace of the cutting plane on the plane of (x, y) makes with the axis of x, we have

$$\tan\phi = \frac{-A}{B},$$

whence
$$\sin\phi = \frac{-A}{(A^2 + B^2)^{\frac{1}{2}}}, \quad \cos\phi = \frac{B}{(A^2 + B^2)^{\frac{1}{2}}}.$$

If the cutting plane do not pass through the origin, we can always make it do so by transferring the axes parallel to themselves. Supposing this to be done, let $OABC$ (fig. 17) be the cutting plane, OC its trace on the plane of (x, y), and $COx = \phi$. Take OC as the new axis of x', and a line Oy' perpendicular to it in the plane $OABC$ as the new axis of y', while the new axis of z' is perpendicular to both Ox' and Oy'. Since, after transforming the co-ordinates to x', y', z', we have to make $z' = 0$ to obtain the intersection of the surface by the plane of (x', y'), or the cutting plane, we may limit ourselves to the consideration of those points alone for which $z' = 0$, that is, for those which lie in the plane $OABC$. Let P be such a point, draw PN, NM parallel to Oz and Oy, and PM' perpendicular to Ox'; then if we join NM' that line is perpendicular to Ox', since Ox' is perpendicular to the plane $PM'N$, and therefore to every line in it. Let

$$OM = x, \quad MN = y, \quad NP = z,$$
$$OM' = x', \quad M'P = y'.$$

Also $PM'N$ is the inclination of the cutting plane to the plane of (x, y), and is therefore represented by θ. Now if we project the broken line $OM'N$ on the line Ox, we have

$$OM = OM' \cos\phi + M'N \sin\phi,$$

and projecting the same line on Oy, we have

$$MN = OM' \sin\phi - M'N \cos\phi.$$

Also $\quad M'N = PM' \cos\theta, \quad PN = PM' \sin\theta,$

therefore

$$\left.\begin{aligned} x &= x' \cos\phi + y' \sin\phi \cos\theta \\ y &= x' \sin\phi - y' \cos\phi \cos\theta \\ z &= y' \sin\theta. \end{aligned}\right\} \quad \ldots\ldots\ldots (11).$$

These values being substituted in the equation to the surface

$$f(x, y, z) = 0,$$

will give an equation $\quad f(x', y') = 0$;

which is the equation to the curve of intersection of the surface and plane.

If the cutting plane be perpendicular to the plane of (x, y), the preceding expressions are reduced to

$$x = x' \cos\phi, \quad y = y' \sin\phi, \quad z = y'.$$

As this transformation of co-ordinates is generally a long and troublesome operation, it is advisable to endeavour to avoid it by having recourse to different methods suited to the problem under consideration.

(69) *If the degree of the surface be* n, *the degree of the curve of intersection cannot be greater than* n.

The equation of the curve of intersection is, as we have seen, found by transforming the co-ordinates till the new plane of (x', y') is parallel to the cutting plane, and then making $z' = 0$. Now, by Art. (67), the degree of the equation to the surface cannot be altered by the transformation of co-ordinates, and hence the new equation in x', y', z' must be of the order n. It is clear, then, that the order of this equation cannot be increased by making $z' = 0$, and therefore the curve of intersection cannot be of a degree greater than n. It may however be less, if it should happen that the vanishing of z' should cause all the powers of the n^{th} degree to disappear.

(70) *Polar co-ordinates.* In the applications of analysis to Mechanics and in the Integral Calculus, polar co-ordinates in space are sometimes found to be useful. The co-ordinates chosen are, usually, the distance of a point from a fixed point or origin, the angle which this distance makes with a fixed axis, and the angle which its projection on a plane perpendicular to the axis makes with a fixed line in the plane. To shew how we may transform from rectangular co-ordinates to these polar co-ordinates, let Ox, Oy, Oz (fig. 18) be the rectangular axes, O' the pole of the polar co-ordinates, of which the co-ordinates are

$$OA = a, \quad AO_1 = \beta, \quad O_1O' = \gamma.$$

Let P be any point, its co-ordinates being x, y, z. Then if we take the axis of z as the fixed axis for the polar co-ordinates, and the axis of x as the fixed line in the plane perpendicular to it, the polar co-ordinates of P are $O'P = r$, the angle between $O'P$ and $Oz = \theta$, and the angle between O_1N and $Ox = \phi$. Hence, from the geometry of the figure, since $O_1N = r \sin \theta$,

$$x = a + r \sin \theta \cos \phi, \quad y = \beta + r \sin \theta \sin \phi, \quad z = \gamma + r \cos \theta. \,.(12),$$

which are the required formulæ of transformation.

CHAPTER IV.

REDUCTION OF THE GENERAL EQUATION OF THE SECOND DEGREE.

(71) In a previous chapter we found that the general equation of the first degree represents only one kind of surface—the plane: our next step is to investigate what kinds of surfaces are represented by the general equation of the second degree. The form of this, when complete in all in its terms, may be written as

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E=0 \;\ldots\; (1),$$

where some of the coefficients are multiplied by 2, for the convenience of future operations.

Since this equation contains ten terms, it is highly desirable, before discussing its geometrical interpretation, to consider whether it may be simplified without destroying its generality. The transformation of co-ordinates gives us the means of trying this, and we proceed to show that we can always, by changing the direction of the co-ordinate axes without altering the origin, effect a very important simplification; and that, by changing the origin and not the direction, we obtain conditions by which we can determine those forms of the equation which offer distinctive peculiarities in their geometrical interpretation.

(72) In the first place, we may put (1) in the shape

$$u_2 + u_1 + u_0 = 0 \;\ldots\ldots\ldots\ldots\; (2);$$

where u_2 is a homogeneous function of the second degree, u_1 of the first degree, and u_0 a constant. Now if we substitute in this equation linear functions, such as we find in the expressions for changing the direction of the co-ordinates, the origin being unaltered, the different terms in (2) must alter independently one of the other, and we may therefore consider them separately.

Taking then the term u_2 alone, we shall show that it may always by transformation of co-ordinates be deprived of the terms involving the products of the variables. For this purpose it may be put under the form

$$u_2=(Ax+B''y+B'z)x+(B''x+A'y+Bz)y+(B'x+By+A''z)z \;\ldots\; (3).$$

Now the formulæ for changing the direction of the co-ordinate axes without altering the origin are, by Art. (60),

$$x = ax_1 + a'y_1 + a''z_1,$$
$$y = bx_1 + b'y_1 + b''z_1,$$
$$z = cx_1 + c'y_1 + c''z_1,$$

the quantities a, b, c &c. being subject to the conditions (3) and (4) of the last chapter. Substituting these values in (3), it takes the form

$$\left.\begin{array}{l} u_2=(Lx_1+L'y_1+L''z_1)(ax_1+a'y_1+a''z_1)+ \\ \quad+(Mx_1+M'y_1+M''z_1)(bx_1+b'y_1+b''z_1)+ \\ \quad+(Nx_1+N'y_1+N''z_1)(cx_1+c'y_1+c''z_1) \end{array}\right\} \ldots\ldots (4);$$

where for shortness we have put

$$L = Aa + B''b + B'c,$$
$$M = B''a + A'b + Bc,$$
$$N = B'a + Bb + A''c,$$

with corresponding values for the accented letters. But (4) may be put under the same form as (3), viz.

$$u_2 = (Px_1 + P'y_1 + P''z_1)x_1+(P_1x_1 + P'_1y_1 + P_1''z_1)y_1$$
$$+ (P_2x_1 + P_2'y_1 + P_2''z_1)z_1 \ldots\ldots (5),$$

where the quantities P are determined by equations of the form

$$P_n^{(m)} = a^{(n)}L^{(m)} + b^{(n)}M^{(m)} + c^{(n)}N^{(m)},$$

the suffix of the P corresponding to the accent on the a, b, c, and the accent on the P to that on the L, M, N. Hence

$$P' = aL' + bM' + cN',$$
$$= a(Aa' + B''b' + B'c') + b(B''a' + A'b' + Bc') + c(B'a' + Bb' + A''c')$$
$$= a'(Aa + B''b + B'c) + b'(B''a + A'b + Bc) + c'(B'a + Bb + A''c)$$
$$= a'L + b'M + c'N = P_1.$$

In like manner we find

$$P'' = P_2, \quad\text{and}\quad P_1'' = P_2'.$$

These six quantities are the coefficients of the products of the variables in (5), and if they be made equal to zero, that equation will be reduced to

$$u_2 = Px_1^2 + P_1'y_1^2 + P_2''z_1^2,$$

which is thus deprived of the terms involving the products.

(73) Now the three conditions

$$P' = P_1 = 0, \quad P'' = P_2 = 0, \quad P_1'' = P_2' = 0 \;\ldots\ldots\; (6)$$

give three relations between the quantities a, b, c &c., and the constants in u_2; and as these quantities are nine in number, and are connected, as we have seen in Arts. (60) and (61), by six equations of condition, we have on the whole nine equations of condition for determining nine quantities; so that unless some of these equations are derivable one from the other, the quantities a, b, c &c. can all be determined. There remains to decide the question whether their values, and also those of P, P_1', P_2'' are possible, in order to prove that the required transformation can be effected. For this purpose let us take the equations

$$\left.\begin{aligned} P_1 &= a'L + b'M + c'N = 0 \\ P_2 &= a''L + b''M + c''N = 0 \\ P &= aL + bM + cN \end{aligned}\right\} \ldots\ldots (7);$$

multiply the first equation by a', the second by a'', and the last by a, and add; then, by the conditions in Art. (62),

$$\left.\begin{aligned} aP = L &= Aa + B''b + B'c, \\ \text{or} \qquad (P - A)\,a &- B''b - B'c = 0. \\ \text{In like manner we find by xg by } b' \text{ \& then by } c' & \\ B''a - (P - A')\,b &+ Bc = 0 \\ B'a + Bb &- (P - A'')\,c = 0 \end{aligned}\right\} \ldots\ldots\ldots (8).$$

Between these we can eliminate a, b, c by cross-multiplication, and we obtain for determining P the equation

$$(P - A)(P - A')(P - A'') - B^2(P - A) - B'^2(P - A') \\ - B''^2(P - A'') - 2BB'B'' = 0 \;\ldots\ldots\; (9).$$

We should arrive at the same equation of condition, if instead of (7) we took the equations

$$P' = 0, \quad P_2' = 0, \quad P_1' = a'L' + b'M' + c'N',$$

and eliminated a', b', c'; or if we took the equations

$$P'' = 0, \quad P_1'' = 0, \quad P_2'' = a''L'' + b''M'' + c''N'',$$

and eliminated a'', b'', c''. Consequently the three roots of the equation (9) are the three quantities P, P_1', P_2''.

(74) This remarkable cubic, which occurs in various Mathematical researches, is of very great importance, as will be seen from the use which we shall make of it; and we shall distinguish it by the name of the *Discriminating Cubic*. But first it is necessary to show that its three roots are always possible, which may be done by the following method due to Cauchy.

In the particular case in which $B' = 0$, $B'' = 0$, the equation (9) is reduced to

$$(P - A)\{(P - A')(P - A'') - B^2\} = 0 \ldots\ldots (10),$$

the roots of which are

$$P = A, \quad P = \tfrac{1}{2}(A' + A'') \pm \tfrac{1}{2}\{(A' - A'')^2 + 4B^2\}^{\frac{1}{2}},$$

all of which are possible, as the quantity under the radical is essentially positive, being the sum of two squares. Now if we put

$$\alpha = \tfrac{1}{2}(A' + A'') + \tfrac{1}{2}\{(A' - A'')^2 + 4B^2\}^{\frac{1}{2}}, \quad \beta = \tfrac{1}{2}(A' + A'') - \tfrac{1}{2}\{(A' - A'')^2 + 4B^2\}^{\frac{1}{2}},$$

we can show that α and β are limits between the roots of the cubic. In the first place we observe that the substitution of these quantities will in virtue of (10) make the first two terms of (9) vanish, and as $\alpha - A'$ and $\alpha - A''$ are essentially positive, since the radical in $\alpha - A'$ exceeds the other term, we may represent them by h^2 and k^2, and as from (10) $B^2 = (\alpha - A')(\alpha - A'')$, the first side of (9) becomes

$$-(B'^2h^2 + B''^2k^2 \pm 2B'B''hk) = -(B'h \pm B''k)^2,$$

which being a negative square is essentially negative. In like manner it may be shown that the substitution of β in the first side of (9) gives a positive result. Consequently, if we write U for the first side of (9), we have the following results: if

$$\begin{array}{ll} P = \infty, & U \text{ is } +, \\ P = \alpha, & U \text{ is } -, \\ P = \beta, & U \text{ is } +, \\ P = -\infty, & U \text{ is } -. \end{array}$$

Therefore one root of the cubic lies between ∞ and α, another between α and β, and the third between β and $-\infty$, and hence the three roots are real.

(75) If any one of these three real roots be substituted in the equations (8), we shall have three equations in a, b, c, which when solved will in general give determinate and possible values for the ratios $\frac{a}{c}$, $\frac{b}{c}$, and these combined with the equation

$$a^2 + b^2 + c^2 = 1,$$

serve to determine the values of a, b, c. These values may be found most readily in the following manner: eliminate c between the first and third, and then between the second and third equations of (8), when we get

$$a\{(P-A)(P-A'')-B'^2\} = b\{B''(P-A'')+BB'\},$$

$$a\{B''(P-A'')+BB'\} = b\{(P-A')(P-A'')-B^2\},$$

whence
$$\frac{a^2}{(P-A')(P-A'')-B^2} = \frac{b^2}{(P-A)(P-A'')-B'^2}$$
$$= \frac{c^2}{(P-A)(P-A')-B''^2},$$

by the symmetry of the formulæ. Let each of these ratios be assumed equal to μ: then

$$a^2 + b^2 + c^2 = 1$$
$$=\mu\{(P-A')(P-A'')+(P-A)(P-A')+(P-A)(P-A'')-B^2-B'^2-B''^2\}.$$

And, on giving to U the same meaning as before, it is obvious that the second side of this equation is equal to $\frac{dU}{dP}$, so that

$$1 = \mu\frac{dU}{dP}.$$

But if P_1, P_2, P_3 be the three roots of the discriminating cubic, and we substitute any one of them as P_1 in $\frac{dU}{dP}$, we have

$$\frac{dU}{dP} = (P_1 - P_2)(P_1 - P_3), \text{ when } P = P_1; \text{ whence}$$

$$\mu = \frac{1}{(P_1 - P_2)(P_1 - P_3)};$$

so that, putting P_1 for P in each of the ratios which is equal to μ, we find

$$a^2 = \frac{(P_1 - A')(P_1 - A'') - B^2}{(P_1 - P_2)(P_1 - P_3)}, \quad b^2 = \frac{(P_1 - A)(P_1 - A'') - B'^2}{(P_1 - P_2)(P_1 - P_3)},$$

$$c^2 = \frac{(P_1 - A)(P_1 - A') - B''^2}{(P_1 - P_2)(P_1 - P_3)}.$$

A similar set of values for $a', b', c', a'', b'', c''$ may be found, by putting P_2 and P_3 for P_1 in the preceding expressions.

(76) It appears then, from the preceding investigation, that since we can always find possible values for the three quantities P, and the nine quantities a, b, c &c. from the conditions (6) Art. (73), the general equation of the second degree may always, without affecting its generality, be reduced to the form

$$Px^2 + P'y^2 + P''z^2 + 2Qx + 2Q'y + 2Q''z + E = 0 \ldots\ldots (11),$$

where for convenience we have dropped the suffixes of the coefficients of the squares of the variables, and put

$$Q = Ca + C'b + C''c, \quad Q' = Ca' + C'b' + C''c', \quad Q'' = Ca'' + C'b'' + C''c''.$$

The only restriction is that the quantities P, P', P'' must not be all equal to zero at the same time, as the equation would then be reduced to the first degree; with this exception these quantities may be of any value or sign.

(77) The separation into different classes of the surfaces represented by (11), depends on the vanishing of one or more of the coefficients of the squares of the variables, as will be seen in the following investigation. If we seek to simplify the equation still further by depriving it of the terms involving the first power of the variables, we change the origin of co-ordinates, putting $\quad x = x_1 + \alpha, \quad y = y_1 + \beta, \quad z = z_1 + \gamma.$

The substitution of these values gives

$$Px_1^2 + P'y_1^2 + P''z_1^2 + 2(P\alpha + Q)x_1 + 2(P'\beta + Q')y_1 + 2(P''\gamma + Q'')z_1$$
$$+ P\alpha^2 + P'\beta^2 + P''\gamma^2 + 2Q\alpha + 2Q'\beta + 2Q''\gamma + E = 0,$$

and the condition that the terms of the first degree shall vanish gives the equations

$$P\alpha + Q = 0, \quad P'\beta + Q' = 0, \quad P''\gamma + Q'' = 0 \ldots\ldots (12)$$

These three equations will give finite and possible values of α, β, γ, in all cases except when any one of the quantities P vanishes. If the corresponding Q be finite, then the value for α, β or γ is infinite: if the corresponding $Q = 0$, the value is indeterminate. Hence we divide the surfaces represented by (1), and also by (11), into the following classes.

I. When none of the quantities P is equal to zero, in which case the equation is reduced to

$$Px^2 + P'y^2 + P''z^2 = H \dots\dots\dots (13).$$

II. When one of the coefficients of the squares as $P = 0$, while Q does not vanish, we cannot make the term involving x vanish, but we may then determine α by the condition that the constant term shall vanish: or in this case,

we may therefore take a third condition that the constant term shall vanish

$$P'\beta^2 + P''\gamma^2 + 2Q\alpha + 2Q'\beta + 2Q''\gamma + E = 0,$$

an equation which must give a possible value for α, since that quantity is involved in the first degree only. The general equation is then reduced to

$$P'y^2 + P''z^2 + 2Qx = 0 \dots\dots\dots (14).$$

The condition of $P = 0$ necessarily involves the condition

$$AB^2 + A'B'^2 + A''B''^2 - AA'A'' - 2BB'B'' = 0 \dots (15),$$

as this is the constant term of the discriminating cubic of which P is a root.

III. When in the equation (11) $P = 0$ and $Q = 0$ at the same time, the other coefficients being finite, it becomes

$$P'y^2 + P''z^2 + 2Q'y + 2Q''z + E = 0,$$

which may be reduced to

$$P'y^2 + P''z^2 = K \dots\dots\dots\dots (16),$$

by changing the origin of co-ordinates.

IV. If we have at the same time $P = 0$ and $P' = 0$, the equation (11) becomes

$$P''z^2 + 2Qx + 2Q'y + 2Q''z + E = 0.$$

If neither Q nor Q' vanish, this equation may always be reduced, as that of Class II. to

$$P''z^2 + 2Qx + 2Q'y = 0 \dots\dots\dots (17).$$

But if $Q = 0$ and $Q' = 0$, the values of α and β are indeterminate, and that of γ alone is determinate. We are then unable

to destroy the constant term, but we may get rid of that involving the first power of z, so that the equation becomes

$$P''z^2 = L \dots\dots\dots\dots\dots\dots (18).$$

The equations $P = 0$, $P' = 0$ involve in addition to (15), the relation $B^2 - A'A'' + B'^2 - AA'' + B''^2 - AA' = 0 \dots\dots (19)$, since two roots of the discriminating cubic in this case vanish. $\frac{dU}{dP}$

(78) Let us now consider the geometrical meaning of this separation: the general equation (11) being

$$Px^2 + P'y^2 + P''z^2 + 2Qx + 2Q'y + 2Q''z + E = 0,$$

let the surface be cut by the line

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r,$$

which passes through a point (a, β, γ). Substituting the values of x, y, z, in terms of r, we have

$$(Pl^2+P'm^2+P''n^2)\,r^2+2\{(Pa+Q)\,l+(P'\beta+Q')\,m+(P''\gamma+Q'')\,n\}r$$
$$+Pa^2+P'\beta^2+P''\gamma^2+2Qa+2Q'\beta+2Q''\gamma+E=0.$$

This being a quadratic equation, gives generally two values of r, which is the length of the portion of the line intercepted between the point (a, β, γ) and the surface. Now the conditions which reduce equation (11) to Class I. make the term involving r disappear, and the quadratic being reduced to two terms gives two values of r which are equal, but of opposite signs. Consequently the point (a, β, γ) bisects every chord in the surface which passes through it. Such a point is called a *centre* of the surface, and the surfaces in Class I. are called *central surfaces*. It is plain, that as the equations (12) give single determinate values for (a, β, γ), there can be only one centre for such surfaces.

Again, the conditions for determining Class II. give an infinite value for a, and finite values for β and γ, since

$$a = -\frac{Q}{0}, \quad \beta = -\frac{Q'}{P'}, \quad \gamma = -\frac{Q''}{P''};$$

therefore these surfaces have their centre at an infinite distance, although they are usually said not to have a centre.

In Class III. we have finite values for β and γ, that of α being indeterminate; hence all the points in the line, of which the equations are

$$\beta = -\frac{Q'}{P}, \quad \gamma = -\frac{Q''}{P}$$

may be considered as centres, as for all these points the coefficient of r vanishes independently of α. Surfaces of this kind are evidently, from Chap. I. Art. (8), cylinders, as their equation involves only two of the variables, and is therefore satisfied independently of the third.

The surfaces represented by Class IV. are of two very different kinds: when neither Q nor Q' vanish, two of the coordinates of the centre are infinite, and the surface consequently has no centre; but when both Q and Q' vanish, the coefficient of r will vanish if

$$P\alpha + Q = 0;$$

and therefore all points in the plane, of which that is the equation, may be considered as centres. Both kinds of surfaces are cylindrical, as will be seen in the following chapter.

CHAPTER V.

INTERPRETATION OF THE EQUATION OF THE SECOND DEGREE.

HAVING reduced the general equation of the second degree to four forms, we now proceed to discuss the geometrical meaning of these equations, and the nature of the surfaces which they represent.

Central Surfaces.

(79) The general equation to these surfaces we found to be

$$Px^2 + P'y^2 + P''z^2 = H \ldots\ldots\ldots\ldots (a);$$

and the different varieties of the surfaces which this equation represents depend on the relative signs of P, P', P'', and the magnitude of H, so that we have four varieties: 1st, when $H=0$; 2nd, when all the quantities P, P', P'' are positive; 3rd, when one of them is negative; 4th, when two of them are negative. All the varieties have this property in common, that they are symmetrical with respect to the origin, since the equation remains unchanged when $-x$, $-y$, $-z$ are put for $+x$, $+y$, and $+z$; and this facilitates their discussion, since we may confine our attention to the absolute positive values of each variable.

(80) *Cones.* 1st, Let $H=0$; then some one at least of the quantities P must be of a different sign from the others, in order that the equation

$$Px^2 + P'y^2 + P''z^2 = 0$$

may represent a surface: for if all be of the same sign, the only possible values of the variables which satisfy the equation are $x=0$, $y=0$, $z=0$, showing that the locus of the equation is in that case a point at the origin. It is sufficient to suppose one only of the coefficients to be negative, as if two be so, we have only to change the sign of the whole equation to bring it

to the other case. Let the equation then be

$$Px^2 + P'y^2 - P''z^2 = 0 \ldots\ldots\ldots\ldots (1).$$

Since this is satisfied by $x = 0$, $y = 0$, $z = 0$, the surface passes through the origin. Let the straight line, of which the equations are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r,$$

meet it in the point x, y, z; then, substituting for x, y, z their values in terms of r, the distance of the point from the origin, we have

$$(Pl^2 + P'm^2 - P''n^2)\, r^2 = 0.$$

This equation can be satisfied only by

$$Pl^2 + P'm^2 - P''n^2 = 0 \ldots\ldots\ldots\ldots (2);$$

and then it is satisfied independently of r, so that that quantity is indeterminate. There being only one relation between l, m, n besides the general one $l^2 + m^2 + n^2 = 1$, there are an infinite number of straight lines, for which the condition (2) is satisfied, and as for all r is indeterminate, all these lines (which pass through the origin) lie wholly in the surface. Such surfaces are called *cones*, the common right cone being a particular case of them.

If we put $z = h$ in (1), it becomes

$$Px^2 + P'y^2 = P''h^2,$$

which is the equation to an ellipse in the plane of (x, y), the origin being at the centre, and the principal axes being parallel to the axes of x and y. Hence all sections made by planes parallel to that of (x, y) are ellipses, which become circles when $P = P'$: it is easy to see that in this case the surface is a right cone. The sections parallel to the planes of (x, z) and (y, z) have for their equations

$$Px^2 - P''z^2 = -P'k^2, \qquad P'y^2 - P''z^2 = -Pk^2,$$

which show that the curves are hyperbolas.

(81) Since an equation of the form

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy = 0,$$

can always, without affecting its generality, be reduced to

$$Px^2 + P'y^2 + P''z^2 = 0,$$

it appears that a *homogeneous* function of the second degree,

when equated to zero, represents in general a cone, the vertex of which is in the origin, unless

1st, the coefficients of the transformed equation are all of the same sign, when it represents a point.

2nd, the function can be decomposed into two possible factors of the first degree, when it represents two planes. The analytical condition that this should be the case is

$$AB^2 + A'B'^2 + A''B''^2 - AA'A'' - 2BB'B'' = 0,$$

since two planes are a particular case of cylinders of the second degree.

(82) *Ellipsoid.* Let H, P, P', P'' be all positive, so that the equation is

$$Px^2 + P'y^2 + P''z^2 = H \quad \ldots\ldots\ldots\ldots (1).$$

Let the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r$$

meet the surface in the point x, y, z, then the combination of (1) and (2) gives

$$(Pl^2 + P'm^2 + P''n^2)\, r^2 = H \quad \ldots\ldots\ldots\ldots (2)$$

as the equation for determining r. Now the coefficient of r^2 can never vanish, since every term in it is essentially positive, consequently r is never infinite, and the surface is therefore a closed surface. Hence if it be cut by any plane, the curve of intersection must be an *Ellipse*, since, by Art. (69), the curve must be of the second degree, and the ellipse is the only closed curve of that degree: from this property the surface is called an *Ellipsoid.*

(83) The ratios of H to P, P', and P'' are quantities which have important geometrical meanings. For let $y = 0$ and $z = 0$ in (1), which is then reduced to

$$Px^2 = H.$$

This determines the distances from the origin at which the axis of x is cut by the surface, which are evidently equal and on opposite sides of the origin. If, then, in fig. (19) we put $OA = OA' = a$, we have

$$a^2 = \frac{H}{P}.$$

In like manner, if $OB = b$, $OC = c$,

$$b^2 = \frac{H}{P'}, \quad c^2 = \frac{H}{P''},$$

and so the equation to the surface may be put in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \ldots\ldots\ldots (3).$$

The lines a, b, c are called the *axes* of the surface, and the points A, B, C its vertices.

It is easy to conclude from this equation that the surface does not extend beyond the points A, A'; for if it be cut by a plane $x=\pm f$, we have

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{f^2}{a^2},$$

an equation which cannot be satisfied by any possible values of y and z if $f>a$, as the second side is then negative. The same holds for the other co-ordinate axes, so that the surface does not extend beyond B and B' along y, and beyond C and C' along z.

(84) If two of the coefficients, as P, P', be equal, which involves the relation $a=b$, then all sections made by planes parallel to (x, y) are circles; for putting $a=b$ and $z=h$ in (3), it becomes

$$\frac{x^2+y^2}{a^2}=1-\frac{h^2}{c^2} \ldots\ldots\ldots (4).$$

The surface in this case is said to be one of revolution round the axis of z, since it may be generated by making an ellipse revolve round one of its axes. If all the quantities P, P', P'' be equal, or $a=b=c$, equation (2) becomes

$$x^2+y^2+z^2=a^2 \ldots\ldots\ldots (5),$$

which shows that every point in the surface is equally distant from the origin, or the surface is a sphere, of which the radius is a. If we change the origin of co-ordinates to an arbitrary point (α, β, γ), this equation becomes

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=a^2 \ldots (6);$$

which is the general equation to a sphere referred to rectangular co-ordinates. Every section of a sphere by a plane is a circle; for all plane sections parallel to the co-ordinate planes are circles, and equation (5) remains unchanged when the axes are transformed to another rectangular system, so that we can thus obtain all possible sections of the surface.

(85) *Hyperboloid of one sheet.* Let one of the coefficients, as P'', be negative, so that the general equation (a) becomes

$$Px^2 + P'y^2 - P''z^2 = H \dots . (1).$$

If we seek the points where this surface is cut by the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r,$$

we find
$$(Pl^2 + P'm^2 - P''n^2)\, r^2 = H \dots . (2).$$

As the coefficient of r^2 may be either positive, zero, or negative, it follows that r may be either real, infinite, or impossible. Consequently the surface extends to infinity in certain directions, which are determined by the condition

$$Pl^2 + P'm^2 - P''n^2 = 0 \dots\dots (3),$$

and no part of it exists in the space for which

$$P''n^2 > Pl^2 + P'm^2.$$

Now l, m, n, being direction-cosines, lie between 0 and 1; and if we suppose $P > P'$, it appears that we cannot have

$$n^2 < \frac{P}{P''}, \quad \text{or} \quad > \frac{P'}{P''}.$$

Therefore the space in which the surface does not extend is that bounded by a surface generated by the straight line r turning round the axis of z, forming with it angles of which the limits are determined by the preceding inequality.

If we cut the surface by planes parallel to the co-ordinate planes, the equations to the sections will be found by putting constants f, g, h in turn for x, y, z in the equation (1), so that we have

$$P'y^2 - P''z^2 = H - Pf^2,$$
$$Px^2 - P''z^2 = H - P'g^2,$$
$$Px^2 + P'y^2 = H + P''h^2.$$

The first two represent hyperbolas, and the third an ellipse; and as they are all possible whatever be the values of f, g, h, it appears that the surface is cut by all planes parallel to the co-ordinate axes, and consequently it is a *continuous* surface of *one sheet.* As the sections parallel to two of the co-ordinate planes are hyperbolas, it is called the *Hyperboloid of one sheet.*

(86) The equation to the surface which limits the surface towards the axis of z may be found by eliminating l, m, n between

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

and

$$Pl^2 + P'm^2 - P''n^2 = 0.$$

On dividing each term of the latter by the square of the corresponding member of the former, the result is

$$Px^2 + P'y^2 - P''z^2 = 0 \ldots\ldots\ldots\ldots (4),$$

the required equation to the limiting surface, which is a cone of the second degree, by Art. (81).

It is easily shown that this cone is an asymptote to the hyperboloid. For, if z' and z be co-ordinates of points in the cone and the surface corresponding to the same values of x and y,

$$P''^{\frac{1}{2}}(z' - z) = (Px^2 + P'y^2)^{\frac{1}{2}} - (Px^2 + P'y^2 - H)^{\frac{1}{2}};$$

or multiplying numerator and denominator of the second side by the sum of the radicals

$$P''^{\frac{1}{2}}(z' - z) = \frac{H}{(Px^2 + P'y^2)^{\frac{1}{2}} + (Px^2 + P'y^2 - H)^{\frac{1}{2}}}.$$

The difference between the co-ordinates z' and z decreases without limit as x and y increase without limit; but z' is always greater than z, so that the cone lies between the axis of z and the surface.

(87) The equation to the hyperboloid may be put in a form similar to the second one of the ellipsoid by introducing corresponding geometrical quantities. Let OA, OA' (fig. 20) be the distances from the origin at which the surface is cut by the axis of x, and put each of them equal to a; let OB, OB', each equal to b, be the corresponding quantities for the axis of y; then

$$a^2 = \frac{H}{P}, \quad b^2 = \frac{H}{P'}.$$

The axis of z never meets the surface, and we cannot assign for it a corresponding geometrical quantity; but if we assume c to be such a quantity that

$$c^2 = \frac{H}{P''},$$

the equation to the surface becomes, by the substitution of these values,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ .... } (5).$$

If $b = a$, or the axes of the ellipse in which the surface is cut by the plane of (x, y) be equal, every section parallel to that plane is a circle, for its equation will be of the form

$$\frac{x^2 + y^2}{a^2} = 1 + \frac{h^2}{c^2},$$

which is that to a circle having its centre on the axis of z, whatever be the value of h, showing that the surface is one of revolution. It may be supposed to be generated by the revolution of an hyperbola round the axis which does not meet it.

If we seek the equation to the asymptotic cone to the hyperboloid under the same form as (5), we find it to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \text{ .... } (6).$$

(88) *Hyperboloid of two sheets.* Let two of the coefficients of the equation (a) be negative, so that it takes the form

$$Px^2 - P'y^2 - P''z^2 = H \text{ .... } (1),$$

then it will be easily seen, by combining this with the equations to the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r,$$

as in the preceding surface, that r is infinite for all values of l, m and n, which satisfy the equation

$$Pl^2 - P'm^2 - P''n^2 = 0 \text{ .... } (2),$$

and consequently that the surface extends to infinity in these directions. And, as before, it may be shown that it is limited by the surface, of which the equation is

$$Px^2 - P'y^2 - P''z^2 = 0, \quad \text{or } P'y^2 + P''z^2 - Px^2 = 0.$$

This is the equation to a cone of which the axis of x is the axis, and which is asymptotic to the surface in such a way that the surface lies between it and the axis of x.

If the surface be cut by planes

$$x = f, \quad y = g, \quad z = h,$$

we have, as the equations to the sections,

$$P'y^2 + P''z^2 = Pf^2 - H,$$
$$Px^2 - P''z^2 = H + Pg^2,$$
$$Px^2 - P'y^2 = H + Ph^2.$$

The first of these is the equation to an ellipse, unless $Pf^2 < H$, in which case it cannot be interpreted; therefore all sections parallel to the plane of (y, z) are ellipses, but no plane drawn at a distance along the axis of x on either side less than $\left(\dfrac{H}{P}\right)^{\frac{1}{2}}$, meets the surface. The surface therefore is discontinuous between the planes so determined. The second and third equations show that all planes parallel to (x, z) and $(\overset{x,y}{\cancel{y, z}})$ cut the surface in hyperbolas, since they are possible, whatever values be assigned to g and h. For these reasons the surface is called the *Hyperboloid of two sheets:* see fig. (21.) If we assume $a^2 = \dfrac{H}{P}$ we see, as in the previous cases, that a is the distance on either side of the origin at which the axis of x is cut by the surface. The other axes never meet the surface; but if we assume

$$b^2 = \frac{H}{P'}, \quad c^2 = \frac{H}{P''},$$

the equation takes the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

and the corresponding equation to the asymptotic cone is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 0.$$

If $P' = P''$, or $b = c$, the sections parallel to (y, z) have for their equation

$$\frac{z^2 + y^2}{b^2} = \frac{f^2}{a^2} - 1,$$

which shows that they are circles, and consequently that the surface is one of revolution. It may be supposed to be generated by the revolution of an hyperbola round its principal axis.

Surfaces without a Centre.

(89) The general equation to these may be put in the form

$$P'y^2 + P''z^2 + Qx = 0, \ldots\ldots\ldots\ldots (b),$$

in which one of the constants may always be supposed to be positive. Moreover the sign of Q can have no influence on the nature of the surface, as the term containing it can always have its sign changed by substituting $-x$ for x, which is equivalent to measuring the positive axis of x in a direction opposite to that before adopted: this will affect the position but not the nature of the surface. Hence there are only two forms of the equation to be considered, one when P' and P'' are of the same sign, the other when they are of contrary signs.

(90) *Elliptic paraboloid.* Taking P' and P'' both positive, and Q negative, the equation is

$$P'y^2 + P''z^2 = Qx \ldots\ldots\ldots\ldots\ldots (1).$$

From the form of the equation it is evident that x can never become negative, and therefore the surface lies wholly on the positive side of the plane of (y, z), while it is symmetrical on opposite sides of the axis of x, since the equation remains unchanged when for y and z we put $-y$ and $-z$. The surface passes through the origin, since $x = 0$, $y = 0$, $z = 0$ satisfy the equation (1). If $z = 0$ we have

$$P'y^2 = Qx, \quad \text{or } y^2 = \frac{Q}{P'}x = px \quad \text{suppose.}$$

This shows that the surface is cut by the plane of (x, y) in a parabola of which p is the principal parameter. In like manner we see that it is cut by the plane of (x, z) in a parabola of which the principal parameter is $p' = \dfrac{Q}{P''}$. On substituting these quantities in the equation, it becomes

$$\frac{y^2}{p} + \frac{z^2}{p'} = x, \quad \text{or } p'y^2 + pz^2 = pp'x \ldots\ldots\ldots (2).$$

If the surface be cut by any plane $x = f$, parallel to (y, z), we have

$$p'y^2 + pz^2 = pp'f,$$

which is the equation to an ellipse, whatever be the magnitude of f, so long as it is positive. Hence all sections parallel to (y, z) on the positive side of the axis of x are ellipses, and since f may be increased indefinitely, the surface extends to infinity in that direction. If the surface be cut by planes parallel to (x, z) and (x, y), $\quad y = g$ or $z = h$,

we have $$pz^2 = p'(px - g^2),$$
and $$p'y^2 = p(p'x - h^2),$$
which are equations to parabolas, of which the latera recta are the same as those of the sections made by the co-ordinate planes of (x, z) and (x, y). For these reasons the surface is called the *Elliptic Paraboloid:* see (fig. 22). When $p = p'$, the elliptic sections parallel to (y, z) become circles, having their centres on the axis of x, so that the surface may be supposed to be generated by the revolution of a parabola round its axis.

(91) *Hyperbolic paraboloid.* Taking P'' as negative in the general form (b), we have to consider the equation

$$P'y^2 - P''z^2 = Qx \ldots\ldots\ldots\ldots\ldots (1).$$

In this case we can assign both positive and negative values without limit to all the variables, and consequently the surface extends indefinitely in all directions. If

$$z = 0, \quad y^2 = \frac{Q}{P'} x = px \text{ suppose,}$$

showing that the plane of (x, y) cuts the surface in a parabola of which p is the principal parameter, and of which the axis is turned towards the *positive* axis of x.

If $$y = 0, \quad z^2 = -\frac{Q}{P''} = -p'x \text{ suppose.}$$

This shows that the plane of (x, z) cuts the surface in a parabola of which the axis is turned towards the *negative* axis of x. Introducing these parameters into the equation, it becomes

$$\frac{y^2}{p} - \frac{z^2}{p'} = x. \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If $x = 0$ we have $P'y^2 - P''z^2 = 0$, which may be decomposed into $$P'^{\frac{1}{2}}y - P''^{\frac{1}{2}}z = 0 \text{ and } P'^{\frac{1}{2}}y + P''^{\frac{1}{2}}z = 0,$$
showing that the plane of (y, z) cuts the surface in two straight lines.

If the surface be cut by a plane $x = f$, parallel to the plane of (y, z), we have $$p'y^2 - pz^2 = pp'f:$$
this is the equation to an hyperbola, whatever be the value of f, positive or negative; but there is a difference between the

sections made on the positive side of the origin and those on the negative side; for in the former the principal axis of the hyperbola is parallel to the axis of y, and in the latter to z. These two kinds of hyperbolic sections are separated by the straight lines in which the surface is cut by the plane of (y, z). As in the previous Article, it is easily seen that all sections made by planes parallel to (x, y) and (x, z) are parabolas, the former having their concavity turned towards the positive axis and the latter towards the negative axis of x. Hence the surface is called the *Hyperbolic Paraboloid:* see (fig. 23).

This surface can never become one of revolution, since the coefficients of y^2 and z^2 can never be the same, as they are essentially of opposite signs.

Surfaces having a Line of Centres.

(92) *Elliptic and hyperbolic cylinders.* It appears from equation (16) of the last chapter that when the coefficients of both the first and second powers of one of the variables vanish, the equation may always be reduced to the form

$$P'y^2 + P''z^2 = K \ldots\ldots\ldots\ldots\ldots (1).$$

As this equation involves only two of the variables, it must, Art. (8), represent a cylindrical surface, such that all the lines drawn parallel to the axis of x which meet the surface lie wholly in it: the particular kind of cylinder depends on the signs of P' and P''. If these be both positive, the trace of the surface on the plane of (y, z) is an ellipse and the surface is called an *Elliptic Cylinder.* If in addition $P' = P''$, the ellipse becomes a circle, and the surface is then a right circular cylinder. If P' and P'' be of opposite signs, the trace on the plane of (y, z) is an hyperbola, and the surface is then called a *hyperbolic cylinder.* It is easy to see that in all cases sections made by planes parallel to (y, z) are the same curve as the trace on that plane; this curve is called the *base* of the cylinder. If the surface be cut by the planes $y = g$, $z = h$ parallel to (x, z) and (x, y), we have

$$P''z^2 = K - P'g^2, \quad P'y^2 = K - P''h^2.$$

Each of these indicates two planes at equal distances from the origin, and hence planes parallel to (x, z), (x, y) cut the surface in two parallel straight lines.

When the equation is in the form

$$P'y^2 + P''z^2 = K,$$

every point in the axis of x is a centre; for if its co-ordinates be $(a, 0, 0)$ the equations to a straight line passing through it are

$$\frac{x-a}{l} = \frac{y}{m} = \frac{z}{n} = r,$$

and where this line meets the surface we have

$$(P'm^2 + P''n^2)\, r^2 = K,$$

which gives two values of r equal but of opposite signs, whatever be the values of l, m, and n, so that the point is a centre. If the equation to the surface be in the form

$$P'y^2 + P''z^2 + 2Q'y + 2Q''z + E = 0 \text{ ...... } (2),$$

the equation to the line of centres may in the same way be shown to be

$$y = -\frac{Q'}{P'}, \quad z = -\frac{Q''}{P''}.$$

The line of centres of a cylindrical surface is called the *axis* of the cylinder.

If in equation (1) $K = 0$ while P' and P'' are of the same sign, the equation indicates a point only; but if P' and P'' be of opposite signs, so that

$$Py^2 - P''z^2 = 0,$$

this may be decomposed into

$$P'^{\frac{1}{2}}y - P''^{\frac{1}{2}}z = 0, \quad \text{and} \quad P'^{\frac{1}{2}}y + P''^{\frac{1}{2}}z = 0,$$

which represent two planes perpendicular to (y, z) and intersecting in the axis of x. These planes are asymptotes to the hyperbolic cylinder, and they may be considered as a particular case of that surface.

Remaining Surfaces of the Second Order.

It appears from the preceding chapter that the remaining surfaces of the second order are included in the two equations

$$P''z^2 + 2Qx + 2Q'y = 0 \text{ .... } (1),$$

$$P''z^2 = L \text{ .... } (2).$$

(93) *Parabolic cylinder.* Let the surface (1) be cut by the straight line

$$\frac{x-a}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,$$

then, putting for x, y, z their values in terms of r,

$$P''n^2r^2+2(P''n\gamma+Ql+Q'm)\,r+P''\gamma^2+2Qa+2Q'\beta=0.$$

Now if (a, β, γ) be a point in the surface, we have

$$P''\gamma^2+2Qa+2Q'\beta=0,$$

and the preceding equation will become indeterminate, if we have, in addition, $n=0,\quad Ql+Q'm=0.$

These conditions are always possible, and therefore a straight line, of which the direction-cosines are determined by the preceding equations, lies wholly in the surface if it meets it at all. As these conditions, joined to $l^2+m^2+n^2=1$, determine one direction only, the straight line which lies wholly in the surface is always parallel to itself, or the surface is a cylinder, and since $n=0$, the cylinder is perpendicular to the axis of z.

If we change the direction of the axes of x and y in their own plane, still keeping them rectangular, by the formulæ

$$x=lx'+my',$$
$$y=mx'-ly';$$

the equation (1) becomes

$$P''z^2+2(Ql+Q'm)x'+2(Qm-Q'l)y'=0\ldots.(3);$$

and if we make the new axis of x' parallel to the generating line of the cylinder, we have

$$Ql+Q'm=0,$$

and the equation (3) takes the form

$$P''z^2+2Ry'=0\ldots\ldots\ldots\ldots\ldots(4).$$

The plane of (z, y') is now perpendicular to the generating line of the cylinder, and the trace of the surface on that plane is a parabola, of which the equation is

$$P''z^2+2Ry'=0;$$

hence the surface is a cylinder with a parabolic base. All the plane sections of this surface are either parabolas or two straight lines, as may be readily seen.

(94) *Two parallel planes.* The equation

$$P''z^2 = L$$

is equivalent to the two

$$P''^{\frac{1}{2}}z - L^{\frac{1}{2}} = 0, \quad P''^{\frac{1}{2}}z + L^{\frac{1}{2}} = 0,$$

which represent two planes parallel to (x, y), and equally distant from the origin on opposite sides. These may be considered as a particular case of cylindrical surfaces, since they may be generated by the motion of a straight line remaining parallel to itself.

We have now discussed the forms of all the varieties of surfaces which are represented by the general equation of the second degree, and in the following chapter we shall demonstrate the more important of their geometrical properties, choosing in preference those which belong to more than one class.

CHAPTER VI.

THEOREMS RELATING TO SURFACES OF THE SECOND ORDER.

Diametral Planes.

(95) We have seen in Art. (78) that any straight line meets a surface of the second order in two points generally; hence any chord, or line bounded by the surface, has a middle point. The surface which passes through the middle points of a series of parallel chords is called a *diametral* surface, and we proceed to shew that for all the surfaces of the second order it is a plane.

(96) Let the equation to the surface be, for shortness, written in the form

$$f(x, y, z) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the symbol f implying a rational function of the second degree. Let the equations to any chord be

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} = r \ldots\ldots\ldots\ldots(2).$$

If we suppose x', y', z' to change while l, m, n remain unaltered, these equations may represent a *system* of chords, all parallel to the line of which the direction-cosines are l, m, n. But if we assume the point (x, y, z) to be in the surface (1), r is the length of the chord between the surface and (x', y', z'). Now from (2) we have

$$x = x' + lr, \quad y = y' + mr, \quad z = z' + nr.$$

Substituting these values in (1), it becomes

$$f(x' + lr, y' + mr, z' + nr) = 0.$$

Expanding by Taylor's Theorem, we have

$$f(x', y', z') + \left(l\frac{df}{dx'} + m\frac{df}{dy'} + n\frac{df}{dz'}\right) r + Rr^2 = 0 \ldots(3),$$

where the terms after the third vanish because the function is of the second degree only, and R is a function of x', y', z', l, m, n. This may be considered as a quadratic equation in r, the two roots of which give the lengths of the two portions of the chord intercepted between the surface and the point (x', y', z'). But if (x', y', z') be the middle point of the chord, the two values of r must be equal, but of opposite signs, and the quadratic equation must be reduced to its first and last term. Hence, when (x', y', z') is the middle point of the chord, we have the condition

$$l\frac{df}{dx'} + m\frac{df}{dy'} + n\frac{df}{dz'} = 0 \ldots\ldots\ldots\ldots (4).$$

Writing now the equation to the surface at full length, it is

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy + 2Cx + 2C'y + 2C''z + E = 0;$$

and therefore $f(x', y', z')$ is a function of the same form, which it may be remarked is not equal to zero, since (x', y', z') is not a point in the surface. Hence

$$\frac{df}{dx'} = 2(Ax' + B'z' + B''y' + C),$$

$$\frac{df}{dy'} = 2(A'y' + Bz' + B''x' + C'),$$

$$\frac{df}{dz'} = 2(A''z' + By' + B'x' + C''),$$

so that equation (4) becomes

$$l(Ax' + B'z' + B''y' + C) + m(A'y' + Bz' + B''x' + C')$$
$$+ n(A''z' + By' + B'x' + C'') = 0 \ldots\ldots (5).$$

Arranging this in terms of x', y', and z', we have

$$(Al + B''m + B'n)x' + (B''l + A'm + Bn)y' + (B'l + Bm + A''n)z'$$
$$+ Cl + C'm + C''n = 0 \ldots\ldots (6).$$

This is a linear relation between x', y', z', and as it remains unchanged so long as l, m, n are constant, it holds equally for the middle points of all the lines for which l, m, n are constant, that is to say all parallel lines; and as it represents a plane, that plane is the locus of the middle points of all the parallel chords of which the direction-cosines are l, m, n, and hence the diametral surface in surfaces of the second order is a plane.

The form (4) is that which is practically most convenient for deducing the equation to the diametral plane from any given form of the equation to the surface. The processes and results of this article hold alike for rectangular and for oblique co-ordinates.

(97) From what has preceded it appears that, if a system of lines be given of which the direction-cosines are l, m, n, we may find the equation to a plane which is the locus of their middle points. To this however there is one exception: if the coefficients of the variables in (6) were each to vanish, so that we had at the same time

$$\left.\begin{aligned} Al + B''m + B'n &= 0 \\ B''l + A'm + Bn &= 0 \\ B'l + Bm + A''n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (7),$$

the plane would be at an infinite distance, since infinite values alone of x, y, z would in that case satisfy equation (6). But in order that the equations (7) may subsist simultaneously, it is necessary that the relation

$$AB^2 + A'B'^2 + A''B''^2 - AA'A'' - 2BB'B'' = 0 \ldots (8)$$

should hold; this relation is easily found by eliminating l, m, n from (7) by cross-multiplication. On comparing (8) with equation (15) of Art. (77), it will be seen that it implies that one of the roots of the discriminating cubic vanishes, and consequently this failure, which is due to the diametral plane being removed to an infinite distance, can occur only in the surfaces without a centre, or those with an infinity of centres as the cylinders.

(98) When the surface has a centre, that point bisects all the chords which pass through it, consequently all diametral planes must pass through the centre; so that if α, β, γ be the co-ordinates of that point, the equation to the diametral plane may be written

$$(Al+B''m+B'n)(x'-\alpha)+(B''l+A'm+Bn)(y'-\beta)+(B'l+Bm+A''n)(z'-\gamma)=0.$$

If we suppose the surface to be referred to the centre, and its equation to be in the form

$$Px^2 + P'y^2 + P''z^2 = H,$$

the equation to the diametral plane is

$$Plx' + P'my' + P''nz' = 0.$$

It is evident that the intersection of any two diametral planes is a diameter of the surface.

If the surface be not central, we may write its equation

$$Py^2 + P'z^2 = 2Qx,$$

and then the equation to the diametral plane is

$$Pmy + P'nz - Ql = 0.$$

This being the equation to a plane perpendicular to (y, z) shows that all the diametral planes are parallel to the axis of x. Hence their intersections are straight lines parallel to the axis of x, and these lines correspond to the diameters in central surfaces.

In the case of cylinders which have a line of centres, the diametral planes all pass through this line.

(99) Conversely, if we have given the equation to a plane, we can find the equation to the line to which are parallel all the chords bisected by the plane. Let

$$Lx + My + Nz = Q \ldots\ldots\ldots\ldots (9)$$

be the equation to the plane, and

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \ldots\ldots\ldots\ldots\ldots (10)$$

the equations to a line passing through the origin and parallel to the chords. The equation to the diametral plane to the system parallel to (10) is, by Art. (96),

$$(Al + B''m + B'n)\,x + (B''l + A'm + Bn)\,y + (B'l + Bm + A''n)\,z$$
$$+ Cl + C'm + C''n = 0.$$

Comparing this with the equation to the given plane, we have

$$L = \lambda\,(Al + B''m + B'n),$$
$$M = \lambda\,(B''l + A'm + Bn),$$
$$N = \lambda\,(B'l + Bm + A''n),$$
$$Q = \lambda\,(Cl + C'm + C''n),$$

from which l, m, n may be determined in terms of L, M, N. If in the equation to the surface we suppose the terms involving the products of the variables to disappear, we have

$$B = 0, \quad B' = 0, \quad B'' = 0,$$

and the preceding equations become

$$L = \lambda A l, \quad M = \lambda A' m, \quad N = \lambda A'' n,$$

and the equation to a diameter parallel to the chords bisected by the given plane are $\dfrac{Ax}{L} = \dfrac{A'y}{M} = \dfrac{A''z}{N}$.

(100) *Definition.* A diametral plane is said to be a *principal* diametral plane when it is perpendicular to the chords which it bisects. We proceed to find whether this geometrical relation be possible.

The direction-cosines of the system of chords being l, m, n, those of the diametral plane are, by Art. (96), proportional to

$$Al + B''m + B'n, \quad B''l + A'm + Bn, \quad B'l + Bm + A''n;$$

therefore the conditions of perpendicularity are, by Art. (45),

$$Al + B''m + B'n = Sl,$$
$$B''l + A'm + Bn = Sm,$$
$$B'l + Bm + A''n = Sn.$$

On comparing these equations with equations (8) of Art. (73), we see that their form is the same, and therefore that the elimination of l, m, n will lead to a cubic equation in S, the same in form as that in P, viz.

$$(S-A)(S-A')(S-A'')-B^2(S-A)-B'^2(S-A')-B''^2(S-A'')-2BB'B''=0,$$

of which the three roots are real, so that there are three ways in which the conditions of principal diametral planes may be satisfied, leading to three sets of values of l, m, n, that is, to three sets of chords which are perpendicular to their diametral planes. The actual values of l, m, n are the same as those of a, b, c in Art. (75), viz.

$$l^2 = \frac{(P_1 - A')(P_1 - A'') - B^2}{(P_1 - P_2)(P_1 - P_3)}, \quad m^2 = \frac{(P_1 - A)(P_1 - A'') - B'^2}{(P_1 - P_2)(P_1 - P_3)},$$

$$n^2 = \frac{(P_1 - A)(P_1 - A') - B''^2}{(P_1 - P_2)(P_1 - P_3)},$$

P_1, P_2, P_3 being the three roots of the discriminating cubic in Art. (73), or of the preceding cubic in S. These values of the direction-cosines of the chords remain determinate even when one of the roots of the cubic vanishes; but in that case the

coefficients of the variables in the equation to the diametral plane are each equal to zero, implying that the plane is removed to an infinite distance. Thus, though the three directions of the principal chords can be always assigned, one of the corresponding diametral planes may not exist. This, it is plain from Art. (97), happens in the surfaces for which

$$AB^2 + A'B'^2 + A''^2B''^2 - AA'A'' - 2BB'B'' = 0.$$

(101) The results at which we have just arrived, compared with those in Art. (73), show us that the process of reducing the general equation of the second degree so as to be deprived of the terms involving the products of the variables, is equivalent to referring the surface to three rectangular axes parallel to the three systems of chords which are perpendicular to their respective diametral planes. Accordingly the geometrical considerations of the properties of diametral planes have been employed by several writers for reducing the general equation: this method was suggested by M. J. Binet, *Correspondance sur l'Ecole Polytechnique*, vol. II. p. 74.

(102) It is to be observed that if we take a diametral plane to be one of the co-ordinate planes, as that of (x, y), and the axis of z to be parallel to its chords, the equation to the surface can contain none but even powers of z. For since the diametral plane bisects the chord parallel to z, the negative values of z must be equal to the positive values, and the equation to the surface must remain unchanged when we substitute $-z$ for z: this cannot happen if the equation contain odd powers of that variable. Conversely, when an equation contains none but even powers of a variable, we know that the plane containing the axes of the other two variables bisects all the chords parallel to the axis of the first variable, or is a diametral plane to chords parallel to that axis. From this it appears that when the equation of the second degree is in the form

$$Px^2 + P'y^2 + P''z^2 = H,$$

the surface being therefore central, each of the co-ordinate planes is a diametral plane of the surface, since the equation contains none but even powers of each of the three variables.

Moreover, since the co-ordinates are supposed to be rectangular, each diametral plane is a *principal* one, and each plane bisects the chords which are parallel to the intersection of the other two.

(103) *Definition.* Three diametral planes are said to be *conjugate* to each other when each bisects the chords which are parallel to the intersection of the other two.

We have just seen that when the equation of the second degree referred to rectangular co-ordinates is in the form

$$Px^2 + P'y^2 + P''z^2 = H,$$

the three co-ordinate planes are *conjugate* diametral planes in the sense just defined, and it is clear that these are the only planes which are at once conjugate and principal planes, since we found before that there are generally only three principal diametral planes in a surface of the second order. We proceed now to shew that there are an infinite number of conjugate diametral planes oblique to their chords, so that when their intersections are taken as oblique axes of co-ordinates, the equation to the surface is reduced to the preceding form.

(104) Taking the equation to a central surface referred to rectangular co-ordinates in the form

$$Px^2 + P'y^2 + P''z^2 = H \qquad (1),$$

let it be cut by a plane passing through the centre,

$$Lx + My + Nz = 0 \qquad (2).$$

Now, by Art. (99), the equations to a line through the centre parallel to the chords bisected by (2), are

$$\frac{Px}{L} = \frac{P'y}{M} = \frac{P''z}{N} \qquad (3).$$

Let

$$L_1x + M_1y + N_1z = 0 \qquad (4)$$

$$L_2x + M_2y + N_2z = 0 \qquad (5)$$

be the equations to two other planes passing through the centre. In order that their intersection may be parallel to the chords bisected by (2), they must both pass through (3): this gives two conditions by Art. (47),

$$\frac{LL_1}{P} + \frac{MM_1}{P'} + \frac{NN_1}{P''} = 0, \quad \frac{LL_2}{P} + \frac{MM_2}{P'} + \frac{NN_2}{P''} = 0.$$

In like manner the conditions that (4) shall bisect the chords parallel to the intersection of (2) and (5) are

$$\frac{LL_1}{P} + \frac{MM_1}{P} + \frac{NN_1}{P''} = 0, \quad \frac{L_1L_2}{P} + \frac{M_1M_2}{P'} + \frac{N_1N_2}{P''} = 0;$$

and the conditions that (5) shall bisect the chords parallel to the intersection of (2) and (4) are

$$\frac{LL_2}{P} + \frac{MM_2}{P'} + \frac{NN_2}{P''} = 0, \quad \frac{L_1L_2}{P} + \frac{M_1M_2}{P'} + \frac{N_1N_2}{P''} = 0.$$

Hence, in order that a system of three oblique conjugate diameters should exist, these six equations must hold: but they are only three independent relations, so that if we suppose L, M, N to be given, there are three relations for determining the four ratios $\frac{L_1}{N_1}, \frac{M_1}{N_1}, \frac{L_2}{N_2}, \frac{M_2}{N_2}$, and consequently one or other of them must be indeterminate. The corresponding plane is therefore indeterminate, and hence we see that if any plane be given, there are an infinite number of pairs of planes which can be drawn so as to be conjugate with it. The intersections of these three planes two and two may be taken as axes of oblique co-ordinates, and there are thus an infinite number of oblique axes, to which when the surface is referred, its equation is reduced to the form

$$Ax^2 + A'y^2 + A''z^2 = K.$$

(105) In the same way as in Art. (83) we see that if a', b', c' be the portions of the axes intercepted between the origin and the surface,

$$a' = \left(\frac{K}{A}\right)^{\frac{1}{2}}, \quad b' = \left(\frac{K}{A'}\right)^{\frac{1}{2}}, \quad c' = \left(\frac{K}{A''}\right)^{\frac{1}{2}},$$

so that the equation to the surface may be written

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1.$$

a', b', c' are called three conjugate diameters, and may, like the principal diameters, be impossible, never meeting the surface.

(106) *To find the relations between oblique conjugate diameters and principal diameters of central surfaces.*

If we refer a central surface to any three conjugate diameters as oblique axes, its equation is of the form

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1 \ldots\ldots\ldots\ldots\ldots (1).$$

We proceed to find an equation for expressing the principal diameters in terms of a', b', c'. The definition of a principal diameter is that it is perpendicular to the diametral plane which bisects the chords parallel to it. Now if

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r \ldots\ldots\ldots\ldots\ldots (2)$$

be the equations to any diameter, the equation to its diametral plane is, by Art. (98),

$$\frac{lx}{a'^2}+\frac{my}{b'^2}+\frac{nz}{c'^2}=0 \ldots\ldots\ldots\ldots\ldots (3).$$

But if f, g, h be the cosines of the angles between the axes of (y, z), (x, z), (x, y) respectively, we have, by Art. (57), as the conditions that (2) should be perpendicular to (3),

$$\left.\begin{aligned} k\frac{l}{a'^2} &= l+hm+gn,\\ k\frac{m}{b'^2} &= hl+m+fn,\\ k\frac{n}{c'^2} &= gl+fm+n, \end{aligned}\right\} \ldots\ldots\ldots\ldots (4),$$

where, by Art. (57), $\quad \dfrac{1}{k}=\dfrac{l^2}{a'^2}+\dfrac{m^2}{b'^2}+\dfrac{n^2}{c'^2}.$

But if we consider x, y, z to be the co-ordinates of the extremity of a principal diameter, their values taken from (2) must satisfy (1). Substituting them, we have

$$\frac{1}{r^2}=\frac{l^2}{a'^2}+\frac{m^2}{b'^2}+\frac{n^2}{c'^2}=\frac{1}{k},$$

where r is the length of a principal semi-diameter. Hence equations (4) become

$$\left.\begin{aligned} \left(\frac{r^2}{a'^2}-1\right)l-hm-gn &= 0,\\ hl-\left(\frac{r^2}{b'^2}-1\right)m+fn &= 0,\\ gl+fm-\left(\frac{r^2}{c'^2}-1\right)n &= 0. \end{aligned}\right\} \ldots\ldots\ldots (5).$$

Eliminating l, m, n between these three equations by cross-multiplication, we find

$$\left(\frac{r^2}{a'^2}-1\right)\left(\frac{r^2}{b'^2}-1\right)\left(\frac{r^2}{c'^2}-1\right)-f^2\left(\frac{r^2}{a'^2}-1\right)-g^2\left(\frac{r^2}{b'^2}-1\right)-h^2\left(\frac{r^2}{c'^2}-1\right)-2fgh=0,$$

a cubic equation in r^2, and therefore furnishing three values, which give the squares of the three principal semi-diameters.

If we arrange this in terms of r^2, it becomes

$$r^6 - r^4(a'^2+b'^2+c'^2)+r^2\{a'^2b'^2(1-h^2)+a'^2c'^2(1-g^2)+b'^2c'^2(1-f^2)\}$$
$$-a'^2b'^2c'^2(1-f^2-g^2-h^2+2fgh)=0 \ldots . (6).$$

The roots of this equation, being the principal semi-diameters, we may call a, b, c, and then the theory of equations gives us the following relations,

$$a^2+b^2+c^2 = a'^2+b'^2+c'^2 \ldots \ldots \ldots \ldots (7),$$

$$a^2b^2+a^2c^2+b^2c^2 = a'^2b'^2(1-h^2)+a'^2c'^2(1-g^2)+b'^2c'^2(1-f^2) \ldots (8),$$

$$abc = a'b'c'(1-f^2-g^2-h^2+2fgh)^{\frac{1}{2}} \ldots \ldots \ldots (9),$$

which are the required relations between the principal diameters and any three conjugate diameters.

Equation (7) signifies, that the sum of the squares of any three conjugate diameters is constant, and equal to the sum of the squares of the principal axes.

Equation (8) shows, that the sum of the squares of the parallelograms formed by each pair of conjugate diameters is constant, and equal to the sum of the squares of the rectangles under each pair of the principal axes.

Equation (9) shows, that the parallelopiped of which the three conterminous edges are conjugate diameters, is constant for all systems of conjugate diameters, and equal to the rectangular parallelopiped, of which the principal axes are diameters.

In equations (7), (8), (9), we have assumed that the quantities a^2, b^2, c^2 are all positive, or that the surface is an ellipsoid; but it is plain that it may be adapted to the other two central surfaces, by changing the sign of one or of two of the quantities a^2, b^2, c^2. In such a case, the corresponding one or two of the quantities a'^2, b'^2, c'^2 will also be negative, so that the equation (9) will still subsist.

(107) In the surfaces without a centre

$$Py^2 + P'z^2 = 2Qx \ldots\ldots\ldots\ldots (1)$$

we said (Art. 98) that all the diametral planes are parallel to the axis, so that their mutual intersections, being parallel straight lines, cannot be taken as a system of co-ordinate axes. We may however find an infinite number of oblique axes for which the equation will be reduced to the preceding form, that is, such that two of the co-ordinate planes shall be diametral and conjugate to the intersection of each other pair. Let

$$My + Nz = K \ldots\ldots\ldots\ldots\ldots (2)$$

be the equation to a diametral plane; then, comparing it with

$$Pmy + P'nz = Ql \ldots\ldots\ldots\ldots (3)$$

which is conjugate with the line $[l, m, n]$, we have

$$Ql = \lambda K, \quad Pm = \lambda M, \quad P'n = \lambda N \ldots\ldots (4),$$

which equations serve to determine the direction of the chords conjugate to (2). A line parallel to this we may take to be one of the new axes, as that of z', drawing it through any arbitrary point in the section made by (2), which we take as the origin in order to get rid of the constant term. The other co-ordinate planes are to pass through this axis of z', and intersect the plane (2) in lines which are to be the axes of x' and y', these being determined by the condition that the equation to the surface shall not contain odd powers of y' or z', or that it shall be of the form

$$Py'^2 + P'z'^2 = 2Qx' \ldots\ldots\ldots\ldots (5).$$

Supposing this to be done, we find, on making $z' = 0$, that the equation to the section by the plane of (x', y') is

$$Py'^2 = 2Qx',$$

which is that to a parabola, referred to an axis and the tangent at its vertex. Hence, as the axis of any parabolic section is parallel to the original axis of x, the three oblique axes which reduce the equation to the form (5) are a line parallel to the axis of the surface drawn through an arbitrary point in the surface, a tangent to the parabolic section made by (2), and the line determined by equations (4). As the point assumed in the section made by (2) is quite arbitrary, it is clear that there are an infinite number of systems of co-ordinates corresponding to every assumed plane.

Similarity of Surfaces.

(108) Two surfaces are said to be similar and similarly placed, when, if we take any arbitrary point O from which radii are drawn to the one surface, we can find another point O' such that the radii drawn parallel to the former and terminated by the other surface are always proportional to them; so that if r_1, r_2, r_3 &c. be radii drawn from O and terminated by one surface, and r'_1, r'_2, r'_3 &c. radii respectively parallel to the former drawn from O' and terminated by the other surface,

$$\frac{r_1}{r'_1} = \frac{r_2}{r'_2} = \frac{r_3}{r'_3} = \&\text{c.} = k.$$

It is to be observed, that if two such points as O and O' can be determined, there are an infinite number of such pairs of points for which the same proposition holds: for if O_1 be any point, the distance of which from O is ρ, and if along the line $O'O_1'$ drawn parallel to OO_1, we assume a distance ρ' such that $\rho = k\rho'$, the triangles, of which two corresponding sides are r, ρ, and r', ρ', are similar, since the angles between r and ρ, r' and ρ' are equal and the sides about them proportional. Hence if r, r' be the radii drawn from O_1 and O_1' to the extremities of r and r', we shall have

$$\frac{r}{r'} = k,$$

or the parallel radii drawn through O_1 and O_1' are in the constant ratio k. Such points are called *centres of similarity.*

(109) *To find the conditions that two surfaces of the second order may be similar.* Let their equations be

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E=0\ldots(1),$$

$$ax'^2+a'y'^2+a''z'^2+2by'z'+2b'x'z'+2b''x'y'+2cx'+2c'y'+2c''z'+e=0\ldots\ldots(2).$$

As the origin is arbitrary we may assume it to be the point O relative to (1), so that the equations to a radius drawn through O are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r \ldots\ldots\ldots\ldots (3);$$

but if α, β, γ be the co-ordinates of O' relative to (2) the equations to a radius parallel to (3) drawn through O' are

$$\frac{x'-\alpha}{l} = \frac{y'-\beta}{m} = \frac{z'-\gamma}{n} = r' \ldots\ldots\ldots\ldots (4).$$

The definition of similarity gives the relation $r' = kr$, which leads to $\quad x' = \alpha + kx, \quad y' = \beta + ky, \quad z' = \gamma + kz.$

If then we substitute these values in (2), the resulting equation in x, y, z must be identical with (1), and on arranging it in powers of these variables, and comparing the coefficients, we shall obtain the required conditions. The substitution leads to

$$k^2(ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy)$$
$$+ 2k\{(a\alpha + b'\gamma + b''\beta + c)x + (a'\beta + b\gamma + b''\alpha + c')y + (a''\gamma + b\beta + b'\alpha + c'')\}$$
$$+ a\alpha^2 + a'\beta^2 + a''\gamma^2 + 2b\beta\gamma + 2b'\alpha\gamma + 2b''\alpha\beta + 2c\alpha + 2c'\beta + 2c''\gamma + e = 0.$$

On comparing the coefficients of the powers of the variables in this equation and in (1), we find

$$\left.\begin{array}{l} \dfrac{a}{A} = \dfrac{a'}{A'} = \dfrac{a''}{A''} = \dfrac{b}{B} = \dfrac{b'}{B'} = \dfrac{b''}{B''} = \\ \dfrac{a\alpha + b'\gamma + b''\beta + c}{kC} = \dfrac{a'\beta + b\gamma + b''\alpha + c'}{kC'} = \dfrac{a''\gamma + b\beta + b'\alpha + c''}{kC''} = \\ \dfrac{a\alpha^2 + a'\beta^2 + a''\gamma^2 + 2b\beta\gamma + 2b'\alpha\gamma + 2b''\alpha\beta + 2c\alpha + 2c'\beta' + 2c''\gamma + e}{k^2E} \end{array}\right\} \ldots\ldots(5).$$

Since the first five equations are independent of α, β, γ and k, it appears that two surfaces cannot be similar, unless the coefficients of the highest powers of the variables be proportional: but to show that the surfaces are actually similar, it is necessary to prove that we can find real values for α, β, γ and k. For this purpose let each of the preceding ratios be put equal to $\dfrac{1}{\lambda}$; the last four ratios give the equations

$$\lambda(a\alpha + b''\beta + b'\gamma) = kC - \lambda c \ldots\ldots (6),$$
$$\lambda(b''\alpha + a'\beta + b\gamma) = kC' - \lambda c' \ldots\ldots (7),$$
$$\lambda(b'\alpha + b\beta + a''\gamma) = kC'' - \lambda c'' \ldots\ldots (8),$$
$$\lambda(a\alpha^2 + a'\beta^2 + a''\gamma^2 + 2b\beta\gamma + 2b'\alpha\gamma + 2b''\alpha\beta + 2c\alpha + 2c'\beta + 2c''\gamma + e) = k^2E \ldots (9).$$

But on combining (9) with (6), (7), (8), multiplied by α, β, γ respectively, we find

$$(kC + \lambda c)\,\alpha + (kC' + \lambda c')\,\beta + (kC'' + \lambda c'')\,\gamma = k^2E - \lambda e \ldots (10),$$

which being linear in α, β, γ may be used instead of (9). These four equations lead to a quadratic equation of two terms for determining k; and if its roots be possible, we must take only the positive one, since k is supposed to be essentially a positive ratio. This single value of k will give corresponding single

values of α, β, γ; and hence there is only one centre of similarity corresponding to that originally assumed.

(110) It is easy to find the geometrical meaning of the conditions

$$\frac{a}{A} = \frac{a'}{A'} = \frac{a''}{A''} = \frac{b}{B} = \frac{b'}{B'} = \frac{b''}{B''} \cdots\cdots (11).$$

For if we suppose the surface (1) to be referred to rectangular co-ordinates parallel to its principal axes, we shall have

$$B = 0, \quad B' = 0, \quad B'' = 0;$$

and therefore, in order that the equations (11) may hold good, we must have

$$b = 0, \quad b' = 0, \quad b'' = 0;$$

or the surface (2) is also referred to its principal diameters, and hence the two surfaces have their principal axes parallel. Moreover if, in addition, (1) be referred to its centre as origin, we should have

$$C = 0, \quad C' = 0, \quad C'' = 0,$$

and therefore the sixth seventh, and eighth, ~~and ninth~~ ratios of (5) give

$$a\alpha + c = 0, \quad a'\beta + c' = 0, \quad a''\gamma + c'' = 0;$$

so that the point (α, β, γ) is, by Art. (78), the centre of the surface (2), and therefore the principal axes of the two surfaces are proportional, since they are corresponding radii. It is obvious also from equations (11), that similar surfaces must be of the same species, since, if one of the quantities A, A', A'' vanish or be negative, the corresponding a, a', a'' must also vanish or be negative, in order that the equations may subsist.

(111) *If two similar surfaces of the second order cut each other, their line of intersection is a plane curve.*

Let the equations to the surfaces be

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E = 0 \ldots (1),$$

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy + 2cx + 2c'y + 2c''z + e = 0 \ldots (2);$$

then the conditions of similarity are

$$\frac{A}{a} = \frac{A'}{a'} = \frac{A''}{a''} = \frac{B}{b} = \frac{B'}{b'} = \frac{B''}{b''} = \lambda.$$

Since the two surfaces intersect, we may combine their equations linearly in any way. Multiply then (2) by λ, and subtract

it from (1); then, by the conditions of similarity, the terms of the second order disappear, and we have

$$2(C-\lambda c)\,x + 2(C'-\lambda c')\,y + 2(C''-\lambda c'')\,z + E - \lambda e = 0 \ldots (3).$$

This is a relation between x, y, z, the co-ordinates of any point of the line of intersection of the surfaces, and as it is of the first degree it represents a plane, so that the line of intersection is a plane curve. As the combination of (1) and (2) leads to only one linear equation, we see that two similar surfaces intersect each other once only.

Since all spheres are necessarily similar surfaces, it appears from this that the line of intersection of two spheres is always a plane curve, and therefore a circle.

(112) *If four similar surfaces intersect each other, the six planes of intersection pass all through one point.*

Let the four equations to the surfaces be

$$u = 0, \quad u_1 = 0, \quad u_2 = 0, \quad u_3 = 0;$$

then, since these may be combined two and two in six different ways, if λ_1, λ_2, λ_3 be the ratios of similarity between the first and each of the others, by the preceding proposition the three equations

$$u - \lambda_1 u_1 = 0, \quad u - \lambda_2 u_2 = 0, \quad u - \lambda_3 u_3 = 0,$$

are the equations to three planes of intersection: and if we eliminate u between each pair of these, the equations

$$\lambda_1 u - \lambda_2 u_2 = 0, \quad \lambda_1 u_1 - \lambda_3 u_3 = 0, \quad \lambda_2 u_2 - \lambda_3 u_3 = 0,$$

are the equations to the other three planes of intersection. Now the first three equations combined together determine the point through which the three planes pass; and since the second three are derived from the first, the values of the co-ordinates derived from combining the first three must satisfy the second three: in other words, the six planes of intersection have one common point.

(113) In connexion with the preceding propositions we may introduce the following:—

If two surfaces of the second degree intersect in a plane curve, their second intersection (when they have one) is also a plane curve.

Let the equations to the surfaces be

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E=0\ldots(1),$$

$$ax^2+a'y^2+a''z^2+2byz+2b'xz+2b''xy+2cx+2c'y+2c''z+e=0\ldots(2).$$

Since the surfaces intersect in a plane, we may take that plane as the plane of (x, y); and therefore, making $z=0$, the equations

$$Ax^2+A'y^2+2B''xy+2Cx+2C'y+E=0\ \ldots\ (3),$$

$$ax^2+a'y^2+2b''xy+2cx+2c'y+e=0\ \ldots\ (4),$$

must be identical, since they are both the equations to the line of intersection. This involves the relations

$$A=\lambda a,\quad A'=\lambda a',\quad B''=\lambda b'',\quad C=\lambda c,\quad C'=\lambda c',\quad E=\lambda e.$$

To find the other intersection, multiply the second equation by λ and subtract it from the first; then, in consequence of the preceding relations, we have

$$(A''-\lambda a'')z^2+2(B-\lambda b)yz+2(B'-\lambda b')xz+2(C''-\lambda c'')z=0\ldots(5).$$

This gives a relation between the co-ordinates of the lines of intersection; and it splits into two linear equations,

$$z=0,$$

and $(A''-\lambda a'')z+2(B-\lambda b)y+2(B'-\lambda b')x+2(C''-\lambda c'')=0.$

The former gives the plane of xy, or the plane of the first intersection; the latter, being of the first degree, is the equation to a plane, and therefore shews that the second intersection is also plane.

If the equation (5) were a complete square, or were reduced to $z^2=0$, it would imply that the two lines of intersection coincide, or that the one surface circumscribes the other, touching it along the plane curve determined by the intersection of the surface with the plane $z=0$.

Of Plane Sections.

(114) The plane sections of a surface of the second order are of course given by combining the equation to the surface with the equation to a plane,

$$\lambda x+\mu y+\nu z=\delta,$$

or $$\lambda(x-a)+\mu(y-\beta)+\nu(z-\gamma)=0\ \ldots\ldots\ (1),$$

if we suppose the plane to pass through a point (a, β, γ). The method of finding the nature of the section indicated in Art (68)

by transforming co-ordinates is necessarily long and tedious, and it is better to avoid it by making use of the distance from a given point to the surface, as that quantity is independent of the co-ordinate axes. Let the equations to a line passing through a point (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \quad \ldots\ldots\ldots \quad (2),$$

where r is the distance between (α, β, γ) and (x, y, z); then if this line r lies in the plane (1), which we suppose to be the plane of section, l, m, n must satisfy the condition

$$\lambda l + \mu m + \nu n = 0 \quad \ldots\ldots\ldots\ldots \quad (3),$$

Hence, instead of combining the equation to the surface with (1), we may find the nature of the section by combining it with (2) and (3), since we shall then have equations for determining all the values of r in the plane section.

(115) To show how this may be done, let us first consider how the species of curves of the second order are discriminated in two dimensions. If the equation to these curves be

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

we know that it represents an ellipse, a parabola, or a hyperbola, according as the function $B^2 - AC$ is negative, zero, or positive. Now if we substitute for x and y their values in terms of r from the equation

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = r,$$

we have $$(Al^2 + 2Blm + Cm^2)\, r^2 + \&c. = 0,$$

and we see that the discriminating condition is equivalent to saying that the curve is an ellipse, a parabola, or a hyperbola, according as the coefficient of r^2, when equated to zero, leads to impossible, equal, or possible values of the ratio $l : m$ or $m : l$. In other words, the curve is a hyperbola when the coefficient of r^2 may be split into two possible and unequal factors, a parabola when it is a complete square, and an ellipse when it cannot be divided into possible factors. This condition is equally applicable in three dimensions.

(116) 1st. For central surfaces: let the equation to the surface be

$$Px^2 + P'y^2 + P''z^2 = H \quad \ldots\ldots\ldots \quad (4).$$

Substituting from (2) in this, we have

$$(Pl^2 + P'm^2 + P''n^2)\, r^2 + \&c. = 0.$$

Now the discriminating condition depends on the nature of the equation

$$Pl^2 + P'm^2 + P''n^2 = 0,$$

combined with

$$l\lambda + m\mu + n\nu = 0;$$

and if between these we eliminate one of the quantities l, m, n, as n, we have

$$(P\nu^2 + P''\lambda^2)\, l^2 + (P'\nu^2 + P''\mu^2)\, m^2 + 2P''\lambda\mu lm = 0,$$

from which we easily find the discriminating condition to be

$$-\nu^2 (PP'\nu^2 + PP''\mu^2 + P'P''\lambda^2),$$

or

$$-\nu^2 \left(\frac{\lambda^2}{P} + \frac{\mu^2}{P'} + \frac{\nu^2}{P''}\right).$$

In the ellipsoid where P, P', P'' are all positive, this is essentially negative, and therefore all the sections are ellipses, as is otherwise apparent. In the cone, and the two hyperboloids, this function may be either negative, zero, or positive, and hence these surfaces may be cut by planes either in ellipses, parabolas, or hyperbolas. The sections will be parabolic when

$$\frac{\lambda^2}{P} + \frac{\mu^2}{P'} - \frac{\nu^2}{P''} = 0,$$

if we consider P'' as the coefficient which is of a sign different from that of the other two.

(117) We may show that this cutting plane is always parallel to some position of the generating line of the asymptotic cone of the hyperboloids, or of the cone itself in that surface. For if l, m, n be the direction-cosines of the generating line, we have, by Art. (80),

$$Pl^2 + P'm^2 - P''n^2 = 0, \quad \text{or } Pl^2 = P''n^2 - P'm^2;$$

but λ, μ, ν being the direction-cosines of the plane of section,

$$\frac{\lambda^2}{P} + \frac{\mu^2}{P'} - \frac{\nu^2}{P''} = 0, \quad \text{or } \frac{\lambda^2}{P} = \frac{\nu^2}{P''} - \frac{\mu^2}{P'}.$$

The former of these equations may for one set of values of l, m, n be satisfied by the system

$$kP^{\frac{1}{2}}l = P''^{\frac{1}{2}}n + P'^{\frac{1}{2}}m, \qquad \frac{1}{k}P^{\frac{1}{2}}l = P''^{\frac{1}{2}}n - P'^{\frac{1}{2}}m,$$

and the latter by the corresponding system

$$-\frac{1}{k}\frac{\lambda}{P^{\frac{1}{2}}} = \frac{\nu}{P'^{\frac{1}{2}}} + \frac{\mu}{P'^{\frac{1}{2}}}, \quad -k\frac{\lambda}{P^{\frac{1}{2}}} = \frac{\nu}{P''^{\frac{1}{2}}} - \frac{\mu}{P'^{\frac{1}{2}}}.$$

Multiplying together the two left-hand and also the two right-hand equations, we have

$$-l\lambda = n\nu + m\mu + \frac{P''^{\frac{1}{2}}}{P'^{\frac{1}{2}}} n\mu + \frac{P'^{\frac{1}{2}}}{P''^{\frac{1}{2}}} m\nu,$$

$$-l\lambda = n\nu + m\mu - \frac{P^{\frac{1}{2}}}{P'^{\frac{1}{2}}} n\mu - \frac{P'^{\frac{1}{2}}}{P^{\frac{1}{2}}} m\nu.$$

Adding them we find

$$l\lambda + m\mu + n\nu = 0,$$

showing that the lines of which the direction-cosines are l, m, n and λ, μ, ν are perpendicular, and therefore that the plane of which the direction-cosines are λ, μ, ν, is parallel to the line $[l, m, n]$. Since k is arbitrary, this is true for all the values of the cosines which satisfy the equations.

(118) 2nd. For surfaces without a centre; their equation is

$$Py^2 + P'z^2 = Qx,$$

so that the equations to be combined are

$$Pm^2 + P'n^2 = 0,$$

$$l\lambda + m\mu + n\nu = 0.$$

Eliminating n between these, we find the discriminating function to be

$$-PP'\frac{\lambda^2}{\nu^2}.$$

When P and P' are of the same sign, that is, in the elliptic paraboloid, this can never be positive, and consequently the surface is never cut by a plane in a hyperbola. The section will be a parabola if $\lambda = 0$, that is, if the plane be parallel to the axis of x. In the hyperbolic paraboloid, when P and P' are of contrary signs, the function is essentially positive, except when $\lambda = 0$, or all plane sections are hyperbolas except those made by planes parallel to the axis of x, which are parabolas.

(119) 3rd. For cylindrical surfaces. The equation to these may be assumed to be

$$Px^2 + P'y^2 + 2Qxy + 2Rx + 2Sy + T = 0,$$

and the equations to be considered are

$$Pl^2 + P'm^2 + 2Qlm = 0,$$

$$l\lambda + m\mu + n\nu = 0,$$

from which, after eliminating m, we find, as the discriminating function,

$$\frac{Q^2 - PP'}{\mu^2};$$

consequently the section is of the same kind as the base of the cylinder of which the discriminating function is $Q^2 - PP'$. Hence a cylinder can be cut by a plane in only one kind of curve; excepting of course when the cutting plane is parallel to the axis of the cylinder, in which case the section consists of two straight lines.

(120) *The sections made by parallel planes in a surface of the second order are all similar curves.*

It is easy to show, by the same method as that used in the case of surfaces, that two curves of the second order are similar, when, if x, y, z be replaced by their values in terms of l, m, n and r, so as to give an equation of the form

$$(Al^2 + 2Blm + Cm^2)\, r^2 + \&c. = 0,$$

the coefficients A, B, C in the two curves are proportional. Now if the equation to the cutting plane be

$$\lambda(x - \alpha) + \mu(y - \beta) + \nu(z - \gamma) = 0 \;\ldots\ldots\ldots\; (1),$$

and those to any radius

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \;\ldots\ldots\ldots\; (2),$$

we must have $\qquad l\lambda + m\mu + n\nu = 0 \;\ldots\ldots\ldots\ldots\; (3).$

But if the equation to the surface be

$$Px^2 + P'y^2 + P''z^2 = H \;\ldots\ldots\ldots\; (4),$$

we must combine (3) with

$$(Pl^2 + P'm^2 + P''n^2)\, r^2 + \&c. = 0 \;\ldots\ldots\; (5).$$

Since then α, β, γ do not appear in the coefficient of r^2, the terms in that function remain the same for all values of α, β, γ, provided λ, μ, ν remain unaltered, that is, for all parallel planes; hence the coefficients of the terms which multiply r^2 are proportional in different positions of the cutting plane, or the curves of section are similar.

(121) *To find the locus of the centres of sections of a surface of the second order made by a series of parallel planes.*

If the surface be central, let its equation be

$$Px^2 + P'y^2 + P''z^2 = H \quad \ldots\ldots\ldots\ldots \text{(1)},$$

and if α, β, γ be the centre of any given section let the equation to the cutting plane be

$$l(x-\alpha) + m(y-\beta) + n(z-\gamma) = 0 \quad \ldots\ldots \text{(2)}.$$

Then $$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu} = r \quad \ldots\ldots\ldots\ldots \text{(3)},$$

subject to the condition

$$l\lambda + m\mu + n\nu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)},$$

are the equations to a line lying in the plane of section and passing through its centre. Hence if we substitute in the equation to the surface the values of x, y, z from (3), we shall obtain a quadratic equation in r, the two roots of which are equal but of opposite signs. Therefore the coefficient of the second term must vanish, which gives the condition

$$P\alpha\lambda + P'\beta\mu + P''\gamma\nu = 0 \quad \ldots\ldots\ldots\ldots \text{(5)};$$

the equation (5) subsists for all values of λ, μ, ν, subject to only one condition (4): hence we have

$$kl = P\alpha, \quad km = P'\beta, \quad kn = P''\gamma,$$

k being an indeterminate multiplier: from which we have

$$\frac{P\alpha}{l} = \frac{P'\beta}{m} = \frac{P''\gamma}{n}$$

as the equations connecting α, β, γ, showing that the locus of the centres is a straight line, in fact the diameter which is conjugate to the central plane section: see Art. (99).

If the surface be not central, its equation is

$$\frac{y^2}{p} + \frac{z^2}{p'} = 2x;$$

instead of equation (5), we have then

$$\frac{\mu\beta}{p} + \frac{\nu\gamma}{p'} - \lambda = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}.$$

Combining this with (4), we have

$$kl = -1, \quad km = \frac{\beta}{p}, \quad kn = \frac{\gamma}{p'};$$

from which we get $\frac{\beta}{p} + \frac{m}{l} = 0, \quad \frac{\gamma}{p'} + \frac{n}{l} = 0$

as the equations to the locus of centres, which is therefore a line parallel to the axis, and hence a diameter.

(122) *To find the axes of a section of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \;\ldots\ldots\ldots\ldots\ldots (1)$$

made by a plane $\quad lx + my + nz = 0 \;\ldots\ldots\ldots\ldots\ldots (2),$

l, m, n being the direction-cosines of the plane.

Since the semi-axes of the elliptical section are the greatest and least radii drawn from the centre to the curve, we must have

$$r^2 = x^2 + y^2 + z^2$$

a maximum or minimum, x, y, z being subject to the conditions (1) and (2). Hence differentiating, the condition for a maximum or minimum gives us

$$xdx + ydy + zdz = 0 \;\ldots\ldots\ldots\ldots (3);$$

with the conditions $\quad \frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2} = 0 \;\ldots\ldots\ldots\ldots (4),$

$$ldx + mdy + ndz = 0 \;\ldots\ldots\ldots\ldots (5),$$

Multiply (4) by an indeterminate multiplier λ, and (5) by μ, and add them to (3); then, equating to zero the coefficients of each differential, we have

$$x + \lambda\frac{x}{a^2} + \mu l = 0$$

$$y + \lambda\frac{y}{b^2} + \mu m = 0$$

$$z + \lambda\frac{z}{c^2} + \mu n = 0.$$

Multiply by x, y, z, and add: then, in virtue of (1) and (2), $r^2 + \lambda = 0$. Hence the equations become

$$\left(1 - \frac{r^2}{a^2}\right)x + \mu l = 0, \quad \left(1 - \frac{r^2}{b^2}\right)y + \mu m = 0, \quad \left(1 - \frac{r^2}{c^2}\right)z + \mu n = 0;$$

whence $\quad x = \frac{\mu a^2 l}{r^2 - a^2}, \quad y = \frac{\mu b^2 m}{r^2 - b^2}, \quad z = \frac{\mu c^2 n}{r^2 - c^2}.$

Multiply by l, m, n, and add: then, in virtue of (2), and dividing by μ, we find
$$\frac{a^2l^2}{r^2-a^2}+\frac{b^2m^2}{r^2-b^2}+\frac{c^2n^2}{r^2-c^2}=0,$$
a quadratic equation for determining r^2. This equation may be adapted to the other central surfaces by changing the sign of one or of two of the quantities a^2, b^2, c^2.

From this expression we can easily determine the area of the section. For the area of an ellipse is equal to the product of the semi-axes multiplied by π: but the last term of the preceding equation (arranged in powers of r^2) is the product of the two roots, that is, of the squares of the two semi-axes. Taking then the square root of this term, and multiplying it by π, we have
$$\frac{\pi abc}{(a^2l^2+b^2m^2+c^2n^2)^{\frac{1}{2}}}$$
as the expression for the area of the section.

Circular Sections.

(123) Since all the surfaces of the second order, except the hyperbolic paraboloid and the hyperbolic and parabolic cylinders, give, when cut by a plane in certain directions, curves which are closed, and consequently must be ellipses, we may enquire whether under any circumstances these sections are circular. And as all parallel sections of a surface of the second order are similar curves, we have only to consider the *direction* of the section, choosing its position in the manner which may be most convenient.

(124) *Central surfaces.* The equation to these is, in general,
$$Px^2+P'y^2+P''z^2=H \ldots\ldots (1),$$
where
$$\frac{H}{P}=a^2,\quad \frac{H}{P'}=b^2,\quad \frac{H}{P''}=c^2.$$

Let the surface be cut by a plane whose equation is
$$z=mx+ny \ldots\ldots\ldots\ldots (2),$$
it being assumed to pass through the origin: we have to determine whether there are any values of m and n which give

a circle as the curve of section. Let the plane (2) also cut a sphere, the equation to which is

$$x^2 + y^2 + z^2 = r^2 \ldots\ldots\ldots\ldots(3);$$

then, if the section of (1) be circular for any values of m and n, we can always assume such a value of r that the section of the sphere, which is always a circle, shall coincide with the section of (1). If then they coincide, their projections on the plane of (x, y) must coincide, and the corresponding equations become identical. The comparison of the several terms of these will give conditions for determining m and n. Substituting then for z in (1) and (3) its value from (2), we get

$$(P + P''m^2)\,x^2 + (P' + P''n^2)\,y^2 + 2P''mn\,xy = H,$$

and $$(1 + m^2)\,x^2 + (1 + n^2)\,y^2 + 2mn\,xy = r^2,$$

as the equations to the projections. As these are to be identical, the coefficients of the several terms must be proportional, and therefore

$$\frac{P + P''m^2}{H} = \frac{1 + m^2}{r^2}, \quad \frac{P' + P''n^2}{H} = \frac{1 + n^2}{r^2}, \quad \frac{P''mn}{H} = \frac{mn}{r^2}.$$

The last condition can be satisfied only by

$$m = 0, \quad \text{or} \quad n = 0.$$

For $m = 0$ we have from the other conditions

$$\frac{H}{P} = r^2, \text{ and } \frac{P' + P''n^2}{H} = \frac{1 + n^2}{r^2} = (1 + n^2)\frac{P}{H};$$

whence $$n = \pm\left(\frac{P - P'}{P'' - P}\right)^{\frac{1}{2}} = \pm\frac{c}{b}\left(\frac{b^2 - a^2}{a^2 - c^2}\right)^{\frac{1}{2}} \ldots\ldots (4).$$

For $n = 0$, we find similarly

$$m = \pm\left(\frac{P' - P}{P'' - P'}\right)^{\frac{1}{2}} = \pm\frac{c}{a}\left(\frac{a^2 - b^2}{b^2 - c^2}\right)^{\frac{1}{2}} \ldots\ldots (5).$$

(125) We proceed to consider how far these values are possible in the different surfaces.

The Ellipsoid. In this all the quantities P are positive, and we shall suppose

$$P < P' < P'', \text{ which is the same as } a > b > c.$$

In this case the formula (4) is impossible; and from (5) we have

$$m = \pm\frac{c}{a}\left(\frac{a^2 - b^2}{b^2 - c^2}\right)^{\frac{1}{2}},$$

which indicates the existence of two series of circular sections, parallel to the planes, of which the equations are

$$a(b^2 - c^2)^{\frac{1}{2}} z - c(a^2 - b^2)^{\frac{1}{2}} x = 0, \quad a(b^2 - c^2)^{\frac{1}{2}} z + c(a^2 - b^2)^{\frac{1}{2}} x = 0.$$

From the form of these it is evident that the two planes pass through the mean axis b, and are perpendicular to the plane containing the greatest and least axes.

If $P = P'$, or $a = b$, the equations to the cutting planes are reduced to $z = 0$, shewing that all the sections parallel to the plane of (x, y), or that containing the two equal axes, are circular, or the ellipsoid is one of revolution round the axis of z. If $P' = P''$, or $b = c$, we have $x = 0$, or the planes parallel to (y, z) give circular sections, and the surface is one of revolution round the axis of x. If $P = P' = P''$, or $a = b = c$, the expressions for m and n are indeterminate, or there are an infinite number of directions in which the surface may be cut by planes in circles. This indeed is obvious, as the ellipsoid then becomes a sphere.

Hyperboloid of one sheet. In this one of the coefficients is negative, as P'', and then the formula (5) gives

$$m = \pm \frac{c}{a}\left(\frac{b^2 - a^2}{b^2 + c^2}\right)^{\frac{1}{2}};$$

and, in order that this may be possible, we must have $b > a$, or the circular sections pass through the greater of the real axes.

To make the surface one of revolution, we can only have $P = P'$, or $a = b$, and there results only one series of circular sections parallel to the plane containing the two equal axes.

Hyperboloid of two sheets. In this case P' and P'' are both negative, and

$$m = \pm \frac{c}{a}\left(\frac{a^2 + b^2}{b^2 - c^2}\right)^{\frac{1}{2}}.$$

In order that this may be possible, we must have $b > c$, or the cutting plane passes through the greater of the two imaginary axes. The surface becomes one of revolution only when $P' = P''$, or $b = c$; and then, as $m = \infty$, the circular sections are perpendicular to the real axis of the surface.

It is to be observed, that the plane which passes through the centre never meets the surface, but planes drawn parallel to it at a sufficient distance cut the surface in circles.

Conical surfaces. The equation to surfaces of this kind are derived from that of the hyperboloid of one sheet, by making $H = 0$; but as this quantity does not enter into the expressions for m, the circular sections of the cone are parallel to those of the hyperboloid, of which it is the asymptote.

Cylindrical surfaces. The elliptic cylinder may be taken as a particular case of the ellipsoid when one of the quantities, as P, vanishes: the value of m then becomes

$$m = \pm \frac{c}{\sqrt{(b^2 - c^2)}},$$

which is possible if $b > c$, or the cutting plane passes through the greater axis of the elliptic base of the cylinder.

(126) *Surface without a centre.* For the elliptic paraboloid the formula is somewhat different. Let the equation to the surface be

$$p'y^2 + pz^2 = pp'x;$$

and let this be cut by a plane

$$x = mz + ny,$$

which also cuts the sphere

$$x^2 + y^2 + z^2 = 2rx.$$

The equations to the projections on the plane of (y, z) are

$$p'y^2 + pz^2 - mpp'z - npp'y = 0,$$

$$(1 + n^2)\, y^2 + (1 + m^2)\, z^2 + 2mn\, yz - 2mrz - 2nry = 0.$$

In order that these may coincide, we must have

$$m = 0 \text{ or } n = 0.$$

If $m = 0$, then $1 + n^2 = \frac{p'}{p}$, and $n = \pm \left(\frac{p' - p}{p}\right)^{\frac{1}{2}}$.

If $n = 0$, then $1 + m^2 = \frac{p}{p'}$, and $m = \pm \left(\frac{p - p'}{p'}\right)^{\frac{1}{2}}$.

If $p' > p$, the first is possible and the second impossible; so that the equation to the cutting plane is

$$p^{\frac{1}{2}}x \mp (p' - p)^{\frac{1}{2}}\, y = 0,$$

giving two sets of circular sections parallel to planes which pass through the axis of z.

If $p > p'$, the second formula is possible, and the equation to the cutting plane becomes

$$p'^{\frac{1}{2}}x \mp (p - p')^{\frac{1}{2}}z = 0,$$

showing that there are two series of circular sections parallel to planes passing through the axis of y.

If $p = p'$, the equation to the cutting plane is

$$x = 0,$$

or the circular sections are all parallel to the plane of (y, z), and the surface is one of revolution round the axis of x.

(127) The following proposition is worthy of note:—*Any two circular sections belonging to different series lie on the surface of the same sphere.*

If the equation to the surface be

$$Px^2 + P'y^2 + P''z^2 - H = 0 \ldots\ldots\ldots\ldots (1),$$

that to the plane of any circular section of one system is

$$(P'' - P')^{\frac{1}{2}}z - (P' - P)^{\frac{1}{2}}x - D = 0 \ldots\ldots\ldots (2),$$

and that to one belonging to the other system is

$$(P'' - P')^{\frac{1}{2}}z + (P' - P)^{\frac{1}{2}}x - D_1 = 0 \ldots\ldots\ldots (3).$$

On multiplying these together, we have

$$(P''-P')z^2-(P'-P)x^2-(D+D_1)(P''-P')^{\frac{1}{2}}z-(D-D_1)(P'-P)^{\frac{1}{2}}x + DD_1 = 0 \ldots\ldots (4),$$

an equation of the second order representing the two cutting planes. Since these planes intersect the surface (1), any equation derived from combining linearly (1) and (4) is the equation to a surface which passes through the intersections of (1) and (4), that is, the circular sections. But on subtracting (4) from (1), we have

$$P'(x^2+y^2+z^2)+(D+D_1)(P''-P')^{\frac{1}{2}}z+(D-D_1)(P'-P)^{\frac{1}{2}}x-DD_1-H=0,$$

which is the equation to a sphere; hence the two circular sections are on the same sphere.

If the equation to the surface be

$$p'y^2 + pz^2 - pp'x = 0 \ldots\ldots\ldots\ldots\ldots (5),$$

those of two planes of circular sections belonging to different systems are

$$p^{\frac{1}{2}}x - (p' - p)^{\frac{1}{2}}y - d = 0 \ldots\ldots\ldots\ldots (6),$$

$$p^{\frac{1}{2}}x + (p' - p)^{\frac{1}{2}}y - d_1 = 0 \ldots\ldots\ldots\ldots (7),$$

whence, as before,

$$px^2-(p'-p)y^2-(d+d_1)p^{\frac{1}{2}}x-(d-d_1)(p'-p)^{\frac{1}{2}}y+dd_1 = 0 \ldots (8).$$

On adding (5) and (8), we find

$$p(x^2+y^2+z^2)-\{(d+d_1)p^{\frac{1}{2}}+pp'\}\,x-(d-d_1)(p'-p)^{\frac{1}{2}}y+dd_1=0,$$

the equation to a sphere on which lie the two circular sections.

(128) From what has preceded, it appears that all surfaces of the second order, except the hyperbolic paraboloid, and hyperbolic and parabolic cylinders, may be generated by the motion of a circle of variable radius which moves so as to be always parallel to one plane. When the surface is of revolution, the plane of the circle is perpendicular to the line of the centres of the sections, but in other cases it is oblique, as may be easily seen; for, by Art. (121), the equation to the line of centres of parallel sections is the diameter conjugate to them, and this can never be perpendicular to their planes, unless it be a *principal* conjugate diameter, which is the case only in surfaces of revolution.

Conditions that the Equation of the Second Degree shall represent Surfaces of Revolution.

(129) When a surface is one of revolution, all the sections made by planes perpendicular to the axis are circles, of which the centres are on the axis; and as any line which, passing through the centre of a circle, bisects a line which does not pass through the centre, is also perpendicular to it, any plane passing through the axis and bisecting a system of parallel chords must be also perpendicular to them. In other words, it is a principal plane to a system of chords perpendicular to the axis. Hence, in surfaces of revolution, there are an infinite number of principal planes, the conjugate chords of which are parallel to one plane; and, conversely, if we investigate the condition that a surface of the second order may admit of an infinite number of principal planes, the chords conjugate to which are parallel to one plane, we shall obtain the condition that it may be a surface of revolution. It is necessary, however, to add that the principal planes are at a finite and determinate distance, since in surfaces of revolution they all intersect in the axis.

(130) The direction-cosines of a system of chords conjugate to a principal diameter are, by Art. (100), given by the equations

$$l^2 = \frac{(P_1 - A')(P_1 - A'') - B^2}{(P_1 - P_2)(P_1 - P_3)}, \quad m^2 = \frac{(P_1 - A)(P_1 - A'') - B'^2}{(P_1 - P_2)(P_1 - P_3)},$$

$$n^2 = \frac{(P_1 - A)(P_1 - A') - B''^2}{(P_1 - P_2)(P_1 - P_3)} \dots\dots\dots\dots (1),$$

P_1, P_2, P_3 being the three roots of the discriminating cubic. In order that there may be an infinite number of such systems of chords, these expressions must become indeterminate, which they will be if both the numerators and denominators of each vanish. The denominators can be made to vanish only by making $P_1 = P_2$ or $P_1 = P_3$, that is, by making two roots of the discriminating cubic equal. The three numerators equated to zero give

$$(P_1 - A')(P_1 - A'') - B^2 = 0, \quad (P_1 - A)(P_1 - A'') - B'^2 = 0,$$
$$(P_1 - A)(P_1 - A') - B''^2 = 0 \dots\dots\dots\dots (2).$$

These equations are not inconsistent with the preceding condition, for it will be seen that they satisfy the equation

$$\frac{dU}{dP} = 0 \dots\dots\dots\dots\dots\dots\dots (3),$$

which is the condition that the discriminating cubic $U = 0$ shall have equal roots. The equations (2) combined with $U = 0$, or

$$(P_1 - A)(P_1 - A')(P_1 - A'') - B^2(P_1 - A) - B'^2(P_1 - A') - B''^2(P_1 - A'') - 2BB'B'' = 0,$$

lead to

$$B^2(P_1 - A) = B'^2(P_1 - A') = B''^2(P_1 - A'') = -BB'B'';$$

whence
$$P_1 = A - \frac{B'B''}{B} = A' - \frac{BB''}{B'} = A'' - \frac{BB'}{B''} \dots\dots (4).$$

These equations give two relations between the coefficients of the general equation of the second degree, which must be satisfied in order that the surface may be of revolution. But we must add the condition that the principal planes corresponding to the chords determined by l, m, n are at a finite distance; this, by Art. (100), is expressed by saying that P_1 shall not vanish, or that the two equal roots of the cubic shall not be zero.

Hence, the condition for a surface being one of revolution may be written in the form

$$A - \frac{B'B''}{B} = A' - \frac{BB''}{B'} = A'' - \frac{BB'}{B''} \gtrless 0 \ldots\ldots (5).$$

(131) Since, by the first of equations (8), of Art. (73), the direction-cosines of the chords are subject to the condition

$$(P_1 - A)\,l - B''m - B'n = 0,$$

the preceding equations (4) change it into

$$B'B''l + BB''m + BB'n = 0 \ldots\ldots\ldots\ldots (6);$$

and if

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

be the equations to a line passing through the origin and parallel to the chords, the elimination of l, m, n between these and equation (6) gives

$$B'B''x + BB''y + BB'z = 0, \quad \text{or } \frac{x}{B} + \frac{y}{B'} + \frac{z}{B''} = 0 \ldots (7)$$

as the equation to a plane parallel to all the chords, and therefore perpendicular to the axis of the surface. To find the equation to this line we have to consider that it is the intersection of all the planes which bisect all chords parallel to (7). Now the equation to the plane which bisects all chords parallel to a line $[l, m, n]$ is, by Art. (96),

$$(Al + B''m + B'n)\,x + (B''l + A'm + Bn)\,y + (B'l + Bm + A''n)\,z$$
$$+ Cl + C'm + C''n = 0 \ldots (8);$$

and when it is also perpendicular to the chord, we have, by Art. (100),

$$Al + B''m + B'n = Sl, \quad B''l + A'm + Bn = Sm, \quad B'l + Bm + A''n = Sn;$$

but, as we have seen, S is given by the same cubic as P, and consequently, instead of S we may put any one root as P_1; by doing so, equation (8) becomes

$$(P_1x + C)\,l + (P_1y + C')\,m + (P_1z + C'')\,n = 0 \ldots (9).$$

The axis of revolution is the line of intersection of all the planes given by this equation, when l, m, n vary, subject to the condition

$$B'B''l + BB''m + BB'n = 0 \ldots\ldots\ldots\ldots (6).$$

As there is only one relation between l, m, n, two of them must be indeterminate, so that if we eliminate one between (6) and

(9), the coefficients of the others must vanish separately, whence we get the equations

$$\frac{P_1x + C}{B'B''} = \frac{P_1y + C'}{BB''} = \frac{P_1z + C''}{BB'};$$

or, putting for P_1 its values from (4), and multiplying by $BB'B''$, we have

$$B\left(x+\frac{BC}{AB-B'B''}\right)=B'\left(y+\frac{B'C'}{A'B'-BB''}\right)=B''\left(z+\frac{B''C''}{A''B''-BB'}\right)\text{...}(10),$$

which are the required equations to the axis, showing it to be perpendicular to the plane (7).

(132) If the general equation of the second degree is deficient in two of the terms involving the products of the variables, that is, if two of the quantities B vanish, the conditions (5) become indeterminate and therefore nugatory. Let the two which vanish be B and B'; then if between the two equations

$$A'' = A - \frac{B'}{B}B'', \quad A'' = A' - \frac{B}{B'}B''\ldots\ldots(11),$$

we eliminate the indeterminate ratio $\frac{B}{B'}$, we have

$$(A'' - A)(A'' - A') = B''^2\ldots\ldots\ldots\ldots(12)$$

as the condition that the surface may be of revolution; but we must add to it that A'' shall not vanish, since when $B = 0$ and $B' = 0$, A'' is the value of P_1.

The equations to the axis of revolution (10) become, in this case, after eliminating from them the ratio $\frac{B}{B'}$ by means of (11),

$$x + \frac{C}{A''} = \frac{A - A''}{B''}\left(y + \frac{C'}{A'}\right), \qquad z + \frac{C''}{A''} = 0\ \ldots.\ (13).$$

If, in addition, $B'' = 0$, or all the terms involving the products of the variables be wanting, the condition (12) becomes

$$A' = A'' \text{ or } A = A''\ \ldots\ldots\ldots\ (14);$$

and from symmetry we may add $A = A'$. Any one of these conditions being satisfied will make the surface be of revolution. Taking the first, the equations to the axis of revolution are

$$A''y + C' = 0, \quad A''z + C'' = 0\ldots\ldots\ldots(15),$$

with similar expressions in the other cases.

Rectilinear Generating Lines.

(133) We have seen before, that cones may be generated by the motion of a straight line which passes constantly through a fixed point; and cylinders by that of a straight line which moves so as to be always parallel to a given position. This suggests the enquiry, whether any of the other surfaces of the second order may be generated in a similar manner. It is evident, *à priori*, that this is impossible for the ellipsoid, since it is a closed curve; for the hyperboloid of two sheets, since it is not a continuous surface; and for the elliptic paraboloid, since it is bounded in one direction. We may therefore confine our attention to the hyperboloid of one sheet, and the hyperbolic paraboloid.

(134) The equation to the former is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \;.............\; (1),$$

which may be written in the form*

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \;.............\; (2).$$

Now this equation may be satisfied by either of the following systems of linear equations:

$$\frac{x}{a} - \frac{z}{c} = k\left(1 - \frac{y}{b}\right), \quad \frac{x}{a} + \frac{z}{c} = \frac{1}{k}\left(1 + \frac{y}{b}\right) \;....\; (A),$$

$$\frac{x}{a} - \frac{z}{c} = k\left(1 + \frac{y}{b}\right), \quad \frac{x}{a} + \frac{z}{c} = \frac{1}{k}\left(1 - \frac{y}{b}\right) \;....\; (B),$$

k being an arbitrary constant. Each of these equations is the equation to a plane, and therefore each system represents a straight line. Since then the equation (1) or (2) may be satisfied by the equations to two straight lines for each value of k, there are two straight lines which lie wholly in the surface represented by (1). As k admits of an indefinite number of values, and to each value of k there corresponds a position of the line in each system, we may, by assigning a proper series of values to k, cause the line represented by either (A) or (B) to trace out the surface (1). Hence there are two ways in which the hyperboloid of one

* This method is due to Bobillier, *Correspondance Mathématique et Physique de Bruxelles*, vol. IV.

sheet may be generated by the motion of a straight line, the one corresponding to equations (A), the other to (B).

(135) It is easy to find the condition to which must be subject the direction-cosines of the generator: for if its equations be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,$$

and we substitute for x, y, z their values in l, m, n and r, the resulting equation in r must be indeterminate, since the line lies wholly in the surface, and therefore the coefficient of each term must vanish separately: that of r^2 gives us the equation

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0 \ldots\ldots\ldots\ldots\ldots (3);$$

hence the generators are all parallel to the generators of the asymptotic cone (Art. 87). From this it appears, that the generator is never in three positions parallel to the same plane; for if this were the case, the three corresponding generators of the cone would lie in one plane, which is clearly impossible, as a cone of the second degree cannot be cut by a plane in more than two straight lines.

(136) There is an important difference between the mode of generation of this surface and that of cones and cylinders. For in the latter, the generating lines either pass through one point or are parallel, and consequently any two lie always in one plane: but in the hyperboloid, if we consider one system of generators as (A), and take any two individuals of the system, as

$$\frac{x}{a}-\frac{z}{c}=k\left(1-\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{k}\left(1+\frac{y}{b}\right),$$

$$\frac{x}{a}-\frac{z}{c}=k'\left(1-\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{k'}\left(1+\frac{y}{b}\right),$$

which may be resolved into the forms

$$2\frac{x}{a}=k+\frac{1}{k}-\left(k-\frac{1}{k}\right)\frac{y}{b},\qquad 2\frac{z}{c}=-\left(k-\frac{1}{k}\right)+\left(k+\frac{1}{k}\right)\frac{y}{b}$$

$$2\frac{x}{a}=k'+\frac{1}{k'}-\left(k'-\frac{1}{k'}\right)\frac{y}{b},\qquad 2\frac{z}{c}=-\left(k'-\frac{1}{k'}\right)+\left(k'+\frac{1}{k'}\right)\frac{y}{b},$$

we find that the condition of their intersection given in equation (5) of Art. (30) leads to the equation

$$(k - k')^2 = 0,$$

which cannot be satisfied so long as k and k' are different: in other words no two lines of the system (A) ever intersect. The same may be shewn of the system (B), and consequently no two generators of the hyperboloid of one sheet are ever in the same plane. Surfaces generated in this manner have been termed by French writers, who first studied their properties, "surfaces gauches": perhaps the nearest equivalent expression in English is "skew surfaces," and that term we shall use for the future. Surfaces which can be generated by the motion of a straight line are called *ruled surfaces*, and are divided into the two classes of *skew surfaces*, of which the hyperboloid of one sheet is the type; and *developable surfaces*, of which the cone may be taken as the type. The reason of the term *developable*, and the nature of the distinction between these two classes of surfaces, will be explained in the chapter on Tangent Planes to Surfaces.

(137) On the other hand it will be readily seen, that every line of the system (A) meets every line of the system (B), since the condition of their intersection leads to an identical equation which is satisfied whatever be the values assigned to k in each equation. This leads to a very simple geometrical mode of regulating the motion of the generating line: for if we take any three lines of the system (B), the motion of a straight line will be completely regulated by constraining it to intersect these three lines. This will appear more clearly from the following considerations: if B_1, B_2, B_3 be any three lines of the system (B), and if at any point in (B_1) we draw two planes, one passing through (B_2) and the other through (B_3), their line of intersection rests both on (B_2) and (B_3), and of course also on (B_1), as the two planes pass through the same point in that line. For each point in (B_1) there is only one such line of intersection, and consequently a line, which is made to pass through any point in (B_1), is completely determined by being constrained to rest on (B_2) and (B_3), so that the motion of a line is completely regulated by being constrained to pass constantly through three given straight

lines. It is to be observed that the three directors (B_1), (B_2), (B_3) must not be all parallel to one plane, since they are taken out of a system of generating lines of the hyperboloid, no three of which, by Art. (135), can be parallel to the same plane. This serves to distinguish this surface from the hyperbolic paraboloid, as we shall see presently. We may of course equally well take any three lines of the system (A) as directors, and constrain the generator to rest always on them. Hence the hyperboloid of one sheet admits of two modes of generation by the motion of a straight line, the directors in the one mode being some of the generators in the other.

(138) Having thus shown that the hyperboloid of one sheet may be generated by the motion of a straight line which rests on three rectilinear directors which do not intersect, and are not all parallel to the same plane, it remains to prove that it is the only surface so generated. For this purpose it is convenient to choose our co-ordinate axes as symmetrically as possible with reference to the three directors, and that is to draw them parallel to these lines, which is always possible as the three directors are supposed to be no two in the same plane. The following construction gives us the means of doing this.

Let B, B_1, B_2 (fig. 24) be the three directors. Through B draw a plane BCD parallel to B_2; through B_1 a plane B_1EF also parallel to B_2. These planes must evidently intersect in a line A_2 parallel to B_2. In like manner through B and B_2 draw planes parallel to B_1, and through B_1 and B_2 planes parallel to B. These six planes form a parallelopiped, at the centre of which, O, we shall place our origin, the axes of x, y and z being parallel to B, B_1 and B_2. Let the sides of three contiguous edges of the parallelopiped be 2α, 2β, 2γ, parallel to x, y, z respectively. Then the equations to the three directors are

$$(B)\begin{cases} y = +\beta, \\ z = -\gamma, \end{cases} \quad (B_1)\begin{cases} z = +\gamma, \\ x = -\alpha, \end{cases} \quad (B_2)\begin{cases} x = +\alpha. \\ y = -\beta. \end{cases}$$

If now the equations to the generating line be

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n},$$

the conditions that it shall pass through (B), (B_1), (B_2) respectively are

$$\frac{y-\beta}{m} = \frac{z+\gamma}{n},$$

$$\frac{z-\gamma}{n} = \frac{x+a}{l},$$

$$\frac{x-a}{l} = \frac{y+\beta}{m}.$$

On multiplying these three equations together, l, m, n are eliminated, and we have

$$(x-a)(y-\beta)(z-\gamma) = (x+a)(y+\beta)(z+\gamma),$$

which may be reduced to

$$ayz + \beta xz + \gamma xy + a\beta\gamma = 0,$$

the equation to the locus. Since this equation is not altered when the signs of the variables are changed, the surface has a centre and is therefore the hyperboloid of one sheet, that being the only central surface of the second order which admits of a rectilinear generator.

The form of the equation will be seen to be analogous to that of the hyperbola referred to its asymptotes; of which the reason is obvious, for the axes, passing through the centre and being parallel to three lines which lie wholly in the surface, are in fact three positions of the generating line of the asymptotic cone.

(139) The equation to the hyperbolic paraboloid is

$$\frac{y^2}{p} - \frac{z^2}{p'} = x,$$

and this may be satisfied by either of the two systems of linear equations

$$\frac{y}{p^{\frac{1}{2}}} - \frac{z}{p'^{\frac{1}{2}}} = k, \quad \frac{y}{p^{\frac{1}{2}}} + \frac{z}{p'^{\frac{1}{2}}} = \frac{1}{k}x \;\ldots\ldots\; \text{(A)},$$

$$\frac{y}{p^{\frac{1}{2}}} + \frac{z}{p'^{\frac{1}{2}}} = k, \quad \frac{y}{p^{\frac{1}{2}}} - \frac{z}{p'^{\frac{1}{2}}} = \frac{1}{k}x \;\ldots\ldots\; \text{(B)};$$

and consequently, as in the case of the hyperboloid, for every value of k there are two straight lines which lie wholly in the

surface. Hence, by assigning all possible values to k, we can obtain from either system a consecutive series of positions of a straight line which lies wholly in the paraboloid: or there are two modes by which this surface may be generated by the motion of a straight line. From the left hand equation of the two systems it appears that the generator is always parallel to a fixed plane

$$p'^{\frac{1}{2}}y - p^{\frac{1}{2}}z = 0, \quad \text{or} \quad p'^{\frac{1}{2}}y + p^{\frac{1}{2}}z = 0,$$

according as it belongs to (A) or (B). This serves to distinguish the surface from the hyperboloid, of which we saw (Art. 135) that no three generators are parallel to the same plane.

(140) It is easy to shew, as in Art. (136), that no two lines of the same system ever intersect, so that the hyperbolic paraboloid is a *skew surface*, but that every line of the one system intersects all the lines of the other. Hence the motion of the generator in one system will be completely regulated if we constrain it to rest constantly on three of the lines of the other system considered as directors: these lines are not arbitrary, but must be taken parallel to one plane, since all the lines in each system (A) and (B) are parallel to one plane. We may also consider the surface as generated by the motion of a straight line which rests constantly on *two* rectilinear directors, while it remains parallel to one plane. These conditions will regulate completely the motion of the generator; for if the two director-lines be cut by a plane parallel to the director-plane, the two points of intersection will determine the position of the generator, and for every parallel plane there is only one such position.

(141) Let us now shew that a line subject to the geometrical conditions of resting on two given straight lines, while it remains parallel to a fixed plane, will trace out the hyperbolic paraboloid.

For convenience we shall use oblique co-ordinates, taking the fixed plane as that of (x, y), the axis of y as passing through the points where the director-lines meet that plane, the origin bisecting the line joining the points, the plane of (x, z) parallel to the director-lines, and the axis of z equally inclined to them. The equations to the directors will then be

$$y = h, \quad x = az \; \ldots\ldots\ldots\ldots \; (1),$$

$$y = -h, \quad x = -az \; \ldots\ldots\ldots \; (2).$$

Since the generating line is parallel to the plane of (x, y) its equations are

$$z = \gamma, \quad \frac{x - a}{\lambda} = \frac{y - \beta}{\mu}.$$

The conditions that it shall pass through (1) and (2) give

$$\frac{x - a\gamma}{\lambda} = \frac{y - h}{\mu} \quad \text{and} \quad \frac{x + a\gamma}{\lambda} = \frac{y + h}{\mu};$$

whence eliminating λ and μ, we have

$$a\gamma y = hx,$$

and as $\gamma = z$, the final equation is

$$ayz = hx.$$

This is of the second degree, and as it involves x and not x^2, it is a surface without a centre; and as it cannot be the elliptic paraboloid, or parabolic cylinder, it must be the hyperbolic paraboloid.

(142) There is a very simple method of constructing practically the hyperbolic paraboloid, which we may here notice. Since through every position of the generating line we may draw a plane parallel to the director-plane, and since parallel planes cut any two lines proportionally, it follows that the generators cut the director-lines proportionally. Consequently, if we take any two finite straight lines not in the same plane, and divide them into the same number of equal parts, lines or threads joining the points of division will form a portion of a hyperbolic paraboloid.

CHAPTER VII.

OF CURVES IN SPACE.

(143) In the first chapter it was shown that a curve considered as existing in space is represented by two equations between the three co-ordinates; these equations being the equations to any two surfaces which, by their intersection, determine the given curve. Also in Art. (44) it was shown that the straight line is the only locus given by the intersection of two planes, that is, of two surfaces of the first degree: in the present chapter we shall briefly consider the nature of curves of a higher order, confining our attention chiefly to those determined by the intersection of the surfaces of the second degree; but we shall first premise some general remarks on curves in space. These are naturally divided into the two classes—those which lie wholly in one plane, or *plane curves,* and those which do not lie wholly in one plane, or *curves of double curvature,* as they are called. The former may always be considered as determined by the intersection of some surface with a plane, and their properties are most easily studied by considering them as existing in two dimensions only; so that it is unnecessary to treat of them here.

(144) It is to be remarked, however, that the most general equations to a plane curve in space are not—the general equation to a surface of the same degree, and the equation to a plane: these would involve too many arbitrary constants. For, as was shown in Art (9), a cylindrical surface may always be supposed to pass through the intersection of any two surfaces: and if we assume the generating line of the cylinder to be parallel to one of the axes, the equation to this cylinder, combined with the equation to a plane, will determine the plane curve in question. Now this cylinder is of the same degree as the given plane

curve, for it is easy to see that all plane sections of a cylinder are of the same degree. Hence, the equations to a plane curve of any degree in space are perfectly general if we combine the equation to a cylindrical surface of the same order, parallel to one of the co-ordinate axes, with the equation to a plane. Thus, in order to obtain the general equation to a plane curve of the second degree, it is not necessary to take the general equation to surfaces of that order containing nine arbitrary constants, and the equation to a plane containing three constants, making twelve in all; but it is sufficient to combine the equation to the plane with that to a cylindrical surface of the second order, which, if it be parallel to one of the co-ordinate axes, contains five constants only: the total number therefore of disposable constants in the equations to a plane curve of the second degree in space is eight only.

(145) It is not difficult to find an analytical condition by which to distinguish between plane curves and those of double curvature. For, as has been said, a plane curve may be considered as represented by the general equation to a plane

$$ax + by + cz = d \quad \ldots\ldots\ldots\ldots \quad (1),$$

combined with the equation to some cylinder which we may write in the form

$$f(x, y) = 0 \quad \ldots\ldots\ldots\ldots \quad (2).$$

These two equations leave one of the variables independent, of which the other two may be considered as functions. If therefore we differentiate equation (1), considering two of the variables as functions of the third, we may, by means of the resulting equations, eliminate the constants which determine the particular plane, and so obtain a relation between x, y, and z which is common to all plane curves. For the sake of symmetry, we may, in differentiating, consider each variable as independent, and introduce the condition of dependency afterwards. After three differentiations we have

$$adx + bdy + cdz = 0,$$
$$ad^2x + bd^2y + cd^2z = 0,$$
$$ad^3x + bd^3y + cd^3z = 0;$$

eliminating a, b, c by cross-multiplication, we find

$$d^3x(dy\,d^2z - dz\,d^2y) + d^3y\,(dz\,d^2x - dx\,d^2z) + d^3z\,(dx\,d^2y - dy\,d^2x) = 0$$

as the required condition. If x be considered as the independent variable of which y and z are functions, $d^2x = 0$, $d^3x = 0$, and the preceding relation becomes

$$\frac{d^3z}{dx^3}\frac{d^2y}{dx^2} - \frac{d^3y}{dx^3}\frac{d^2z}{dx^2} = 0.$$

(146) The intersection of two surfaces of the second degree is in general a line of which the projection on any plane is a curve of the fourth degree. To this proposition, however, there are exceptions which we shall consider.

When two surfaces of the second order have a common principal plane, the line of their intersection is projected on this plane in a curve of the second degree. If we assume the plane of (x, z) to be the principal plane common to the two surfaces, their equations cannot contain odd powers of y, and will therefore be of the form

$$Ax^2 + A'y^2 + A''z^2 + 2B'xz + 2Cx + 2C''z + E = 0,$$

$$ax^2 + a'y^2 + a''z^2 + 2b'xz + 2cx + 2c''z + e = 0;$$

if we multiply the first by a', and the second by A', and subtract, we have

$$(Aa' - A'a)\,x^2 + (A''a' - A'a'')\,z^2 + 2\,(B'a' - A'b')\,xz$$
$$+ 2\,(Ca' - A'c)\,x + 2\,(C''a' - A'c'')\,z + Ea' - A'e = 0,$$

which does not contain y, and is therefore the equation to the projection of the line of intersection on the plane of (x, z) or the principal plane, and is of the second degree.

This proposition evidently includes the intersection of similar surfaces, which by Art. (111) is a plane curve.

(147) The projection of the intersection of two surfaces of the second degree may be a curve of the third degree, as is seen in the following remarkable proposition, due to M. Quetelet.* *All plane curves of the third degree are the projections of the curves of intersection of surfaces of the second degree.* The general equation of the third degree in two variables is

$$ax^3 + bxy^2 + b_1x^2y + a_1y^3 + cx^2 + dxy + c_1y^2 + ex + e_1y + f = 0 \ldots (1).$$

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Now if we change the direction of the co-ordinates by the formulæ

$$x = lx_1 - my_1, \qquad y = mx_1 + ly_1,$$

the coefficients of the terms of the third degree in the transformed equation will involve l and m in the third degree, and therefore any one of them equated to zero must give a possible value for the ratio $l : m$; that is, it is always possible to transform the co-ordinates so as to deprive the equation (1) of one of the terms of the third degree. Let this be the term involving y^3; then the equation will be in the form

$$x(ax^2 + by^2 + b_1xy) + cx^2 + dxy + c_1y^2 + ex + e_1y + f = 0 \dots\dots (2).$$

This evidently may be considered as the result of the elimination of z between

$$ax^2 + by^2 + b_1xy = z \dots\dots\dots\dots (3),$$

and

$$cx^2 + dxy + xz + c_1y^2 + ex + e_1y + f = 0 \dots\dots (4).$$

But (3) and (4) are the equations to two surfaces of the second degree, and (2) is the equation to the projection on (x, y) of their curve of intersection; consequently all the curves represented by (2), and hence by (1), or all plane curves of the third degree, may be considered as the projections of the intersection of two surfaces of the second degree. We may remark that the equation (3) may, by changing the co-ordinates x and y in their own plane, without altering z, be put in the form

$$Ax^2 + By^2 = z,$$

from which we see that it represents one of the paraboloids, the axis of the surface being perpendicular to the plane of projection.

(148) Moreover the curve of the fourth degree, in which is projected the intersection of two surfaces of the second degree, may sometimes be split into two equations of lower degrees. Thus if, as in the theorem of Art. (113), the two intersections are plane curves, the curve of the fourth degree may be divided into two curves of the second degree. In such a case one of the surfaces of the second degree may be replaced by a system of two planes, which by Arts. (92) and (94) may be considered as a particular case of surfaces of the second order.

(149) If the equation to a surface of the second order be given, it is easy to assign the equation to the surface which shall intersect it in two given planes. For if

$$u_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the equation to the surface, and

$$u_1 = 0, \quad v_1 = 0 \ldots\ldots\ldots\ldots\ldots (2)$$

be the equations to the given planes, the equation

$$u_2 + \lambda u_1 v_1 = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

λ being any constant, is the equation to the surface required. This is easily seen on combining equations (1) and (3) by subtraction, for we then get $\lambda u_1 v_1 = 0 \ldots\ldots\ldots\ldots\ldots (4)$, which is satisfied either by

$$u_1 = 0, \quad \text{or by} \quad v_1 = 0;$$

that is, the planes represented by these equations pass through the intersections of the surfaces represented by (1) and (3). If the surfaces, instead of intersecting in two plane curves, touch each other along one plane curve, the equations to the two planes must become the same, or $u_1 = v_1$. Hence the equation

$$u_2 + \lambda u_1^2 = 0 \ldots\ldots\ldots\ldots\ldots (5)$$

is the equation to a surface of the second order which is inscribed in, or circumscribed about, the surface of which the equation is

$$u_2 = 0,$$

the equation to the plane of contact being

$$u_1 = 0.$$

Again, if $v_1 = 0$ be the equation to a plane in which the surface (1) is touched by some other surface of the second order, the equation to the latter is

$$u_2 + \mu v_1^2 = 0 \ldots\ldots\ldots\ldots\ldots (6).$$

Now if we suppose the surfaces (5) and (6) to intersect, we have, on combining their equations by subtraction,

$$\lambda u_1^2 - \mu v_1^2 = 0 \ldots\ldots\ldots\ldots\ldots (7),$$

which is satisfied by

$$\lambda^{\frac{1}{2}} u_1 - \mu^{\frac{1}{2}} v_1 = 0, \quad \text{or} \quad \lambda^{\frac{1}{2}} u_1 + \mu^{\frac{1}{2}} v_1 = 0 \ldots\ldots (8),$$

and these, being linear equations, represent two planes. Hence it appears that if two surfaces of the second order be inscribed in, or circumscribed about, another surface of that order, their lines of intersection, when they cut each other, are plane curves.

(150) Again, if two ruled surfaces of the second order intersect along a generating line common to both, the curve of intersection will be projected on any plane in a straight line which is of the first degree, and another curve which must be in general one of the third degree. Thus, for example, if the cone of which the equation is

$$z^2 - m^2 (x^2 + y^2) = 0$$

be cut by the cylinder of which the equation is

$$z^2 + m^2 (x^2 + y^2) - 2mxz + maz - m^2a (x + y) = 0,$$

one line of intersection is the straight line of which the equations are

$$y = 0, \quad z - mx = 0,$$

and which is projected on the plane of (x, y) along the axis of x, while the other line of intersection is projected on the plane of (x, y) in a curve of the third order of which the equation is

$$2(y - a)(x^2 + y^2) + a^2x = 0.$$

(151) After plane curves the most interesting class consists of those which can be drawn on the surface of a sphere. These are of course determined by the intersection of a sphere with some other surface depending on the nature of the curve in question; the equation to the surface being deduced from the definition of the curve. The best method of studying the properties of such curves is however, not by referring them to three co-ordinates in space, but to two curvilinear co-ordinates on the surface of the sphere; a method closely resembling the co-ordinate geometry of plane curves. It would be out of place here to explain this method, and it will be sufficient to refer the reader to the original memoirs on the subject, which are by Gudermann in *Crelle's Journal*, Band. VI. and XIII.; Davies in the *Edinburgh Transactions*, vol. XII.; and Graves in an Appendix to a *Translation of two Memoirs by M. Chasles*: this last work in particular I would recommend, as the method there pursued is the most symmetrical and elegant. In the present place I shall content myself with shewing how, from the definitions of some of these curves, we may deduce the equations to the surfaces which intersect the sphere.

(152) *To find the equations to the equable spherical spiral.*

This curve is defined in the following manner:—If a meridian PRP' (fig. 25) on a sphere revolve uniformly about an axis PP', which is a diameter of the sphere, while a point M moves uniformly along the meridian from P to P', so as to describe an arc on the meridian equal to the angle through which the meridian has revolved, the locus of M is the spiral in question.

Taking the axis PP' as that of z, PAP', the initial position of the plane of the meridian, as the plane of (x, z), the equation to the sphere is
$$x^2 + y^2 + z^2 = r^2 \ldots\ldots\ldots\ldots\ldots(1).$$
Now let $POM = \theta$, $AON = \phi$, then, by the definition of the curve, $\theta = \phi$. But we have generally, by Art. (70),
$$x = r\cos\theta\cos\phi, \quad y = r\cos\theta\sin\phi;$$
and in this case
$$x = r\cos^2\theta, \quad y = r\cos\theta\sin\theta.$$
Hence
$$x^2 + y^2 = r^2\cos^2\theta\,(\cos^2\theta + \sin^2\theta) = rx.$$
Therefore the surface defined by the equation
$$x^2 + y^2 - rx = 0 \ldots\ldots\ldots\ldots\ldots(2)$$
by its intersection with the sphere determines the spiral in question. It is easy to see that this second surface is a right circular cylinder perpendicular to the plane of (x, y), the diameter of its base being the radius of the sphere. If we wish to represent the curve by two equations involving each two variables only, we may subtract the second equation from the first. We have then
$$z^2 = r(r - x) \ldots\ldots\ldots\ldots\ldots(3),$$
which is the equation to a parabolic cylinder perpendicular to the plane of xz. Hence the curve may be considered as determined by the intersection of a right circular and a right parabolic cylinder at right angles to each other.

Again, if we eliminate x between (1) and (3) we obtain, as the equation to a cylindrical surface perpendicular to the plane of (y, z),
$$z^4 = r^2(z^2 - y^2) \ldots\ldots\ldots\ldots\ldots(4),$$
which is of the fourth order. Hence the equable spherical spiral may be represented by any one of the following systems

of equations to cylinders:

$$x^2 + y^2 = rx, \qquad z^2 = r(r - x,)$$
$$x^2 + y^2 = rx, \qquad z^4 = r^2(z^2 - y^2),$$
$$r(r - x) = z^2, \qquad z^4 = r^2(z^2 - y^2),$$

or by the combination of the equation to the sphere with any one of them.

(153) *To find the equations to a spherical ellipse.*

The spherical ellipse is a curve traced on the surface of a sphere such that the sum of the distances of any point from two fixed points is constant. Let S, H (fig. 26) be the two fixed points on the surface of the sphere, C the middle point between them. If P be any point in the spherical ellipse, SP, HP arcs of great circles, then the definition gives us

$$SP + HP = \text{a constant} = 2\alpha \text{ suppose.}$$

Through P draw PM, an arc of a great circle perpendicular to SH, and let $SH = 2\gamma$, $CM = \phi$, $PM = \theta$. Then, in the right-angled triangle SPM, we have by Napier's rules

$$\cos SP = \cos(\gamma - \phi)\cos\theta.$$

In like manner from the triangle HPM

$$\cos HP = \cos(\gamma + \phi)\cos\theta.$$

Now $\quad \cos SP + \cos HP = 2\cos\dfrac{(SP + HP)}{2}\cos\dfrac{(SP - HP)}{2},$

$$\cos HP - \cos SP = 2\sin\frac{(SP + HP)}{2}\sin\frac{(SP - HP)}{2},$$

and $SP + HP = 2\alpha$; therefore, after reduction, we find

$$\cos\tfrac{1}{2}(SP - HP) = \frac{\cos\gamma\cos\phi\cos\theta}{\cos\alpha},$$

$$\sin\tfrac{1}{2}(SP - HP) = \frac{\sin\gamma\sin\phi\cos\theta}{\sin\alpha};$$

squaring and adding, we find

$$\left(\frac{\cos^2\gamma}{\cos^2\alpha}\cos^2\phi + \frac{\sin^2\gamma}{\sin^2\alpha}\sin^2\phi\right)\cos^2\theta = 1.$$

Now if we take OA as the axis of x, OC as that of y, O being the centre of the sphere, and call r the radius of the sphere, we have, by Art (70),

$$x = r\cos\phi\cos\theta, \qquad y = r\sin\phi\cos\theta;$$

so that the preceding equation is equivalent to

$$\frac{\cos^2\gamma}{\cos^2\alpha}x^2 + \frac{\sin^2\gamma}{\sin^2\alpha}y^2 = r^2 \quad \ldots\ldots\ldots \text{(1)}.$$

This is the equation to a right elliptical cylinder perpendicular to the plane of (x, y), and, being combined with the equation to the sphere

$$x^2 + y^2 + z^2 = r^2 \ldots\ldots\ldots\ldots\ldots \text{(2)},$$

it determines the spherical ellipse.

If we subtract (1), multiplied by $\sin^2\alpha$, from (2), we have

$$(1 - \tan^2\alpha\cos^2\gamma)\, x^2 + \cos^2\gamma\, y^2 + z^2 = r^2\cos^2\alpha \ldots\ldots \text{(3)},$$

which is the equation to an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots \text{(4)},$$

if $\quad c^2 = r^2\cos^2\alpha, \quad b^2 = r^2\dfrac{\cos^2\alpha}{\cos^2\gamma}, \quad a^2 = \dfrac{r^2\cos^2\alpha}{1 - \tan^2\alpha\cos^2\gamma}.$

Hence the spherical ellipse may be considered as the intersection of a sphere with a concentric ellipsoid. Its equations may also be exhibited in a symmetrical form analogous to those of the straight line. For if we eliminate z^2 between (2) and (4), we have

$$x^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + y^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) = \frac{r^2}{c^2} - 1.$$

But if f, g be two corresponding values of x and y,

$$f^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + g^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) = \frac{r^2}{c^2} - 1,$$

and hence by subtraction and obvious reduction

$$\frac{x^2 - f^2}{a^2(c^2 - b^2)} = \frac{y^2 - g^2}{b^2(a^2 - c^2)} = \frac{z^2 - h^2}{c^2(b^2 - a^2)} \ldots\ldots\ldots \text{(5)},$$

by the symmetry of the formulæ.

(154) After curves given by the intersection of surfaces of the second degree, there are scarcely any of interest except the *helix*, the equations to which we proceed to find.

The *helix* is a curve traced on a right circular cylinder, in such a way that the co-ordinate parallel to the generating line of the cylinder is proportional to the arc of the circular base intercepted between the foot of the ordinate and a fixed point.

Taking the centre of the circular base of the cylinder as origin, the axis of the cylinder as the axis of z, and making the axis of x pass through the fixed point in the circular base, and calling s the intercepted portion of the circular arc, the definition of the helix gives us the relation

$$z = ks,$$

k being the coefficient of proportionality. But if a be the radius of the circular base of the cylinder,

$$x = a \cos \frac{s}{a}, \quad y = a \sin \frac{s}{a},$$

therefore
$$x = a \cos \frac{z}{ka}, \quad y = a \sin \frac{z}{ka};$$

these two equations taken together are the equations to the curve; but it may also be expressed by either of these combined with the equation to the cylinder

$$x^2 + y^2 = a^2.$$

Since $\cos \dfrac{z}{ka} = \cos\left(2n\pi + \dfrac{z}{ka}\right)$, and $\sin \dfrac{z}{ka} = \sin\left(2n\pi + \dfrac{z}{ka}\right)$, n being any integer, it appears that the same values of x and y correspond to an infinite number of values of z, or the ordinate z meets the curve in an infinite number of points. These points are separated from each other by an interval $2\pi ka$; and if we call this h, we have

$$k = \frac{h}{2\pi a},$$

and the equations to the curve may be put in the form

$$x = a \cos\left(\frac{2\pi z}{h}\right), \quad y = a \sin\left(\frac{2\pi z}{h}\right).$$

These equations show that the helix is projected on the planes of (x, z) and of (y, z) as the curve of sines, while it is obvious that it is projected on the plane of (x, y) as a circle.

CHAPTER VIII.

OF TANGENTS AND NORMALS TO SURFACES.

(155) *Definition.* If through any point in a surface a straight line be drawn, meeting the surface again in at least one other point; and if, as the second point moves up to the first along any given curve traced on the surface, there be a limiting position of the cutting line, the line in that position is called a *Tangent Line* to the surface. Since an infinite number of curves passing through one point may be traced on the surface, there are an infinite number of tangent lines which can be drawn at any given point. We therefore cannot determine the equations to any one line, but we may find the condition to which must be subject the constants in the equations to all lines which are tangents at the given point.

(156) Let the equation to the surface be

$$F(x, y, z) = 0 \dots\dots\dots\dots\dots (1),$$

and the equations to any line passing through a point (x, y, z)

$$\frac{x'-x}{l} = \frac{y'-y}{m} = \frac{z'-z}{n} = r \dots\dots\dots\dots (2),$$

x', y', z' being the current co-ordinates of the line. Then if x_1, y_1, z_1 be the co-ordinates of the point nearest to (x, y, z), in which the line meets the surface again, they must satisfy the equation to the surface (1) as well as those to the line (2), so that we have

$$F(x_1, y_1, z_1) = 0 \dots\dots\dots\dots\dots (3),$$

and $\quad x_1 = x + lr, \quad y_1 = y + mr, \quad z_1 = z + nr \dots\dots (4),$

r being the length of the chord between the point (x, y, z), and (x_1, y_1, z_1). Substituting the values of x_1, y_1, z_1 from (4) in (3), and expanding by Taylor's theorem, we find

$$F(x, y, z) + \left(l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz}\right)r + Rr^2 = 0 \dots (5);$$

where R is a function of x, y, z and positive powers of r. But as (x, y, z) is a point in the surface, the first term of (5) vanishes and the equation becomes

$$r\left(l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}+Rr\right)=0 \quad \ldots\ldots\ldots (6).$$

This is satisfied either by $r = 0$, or by

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}+Rr=0 \quad \ldots\ldots\ldots (6').$$

The former of these merely gives the point (x, y, z); the second is an equation for determining r, and therefore one of the other points in which the line (2) meets the surface. But if, as we assumed, r correspond to the point nearest to (x, y, z), and if we suppose this point to move up to (x, y, z), r diminishes without limit, and by our definition the straight line becomes ultimately a tangent. But unless R become infinite (which case we do not here consider), the limit of the equation (6′) is

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}=0 \quad \ldots\ldots\ldots (7),$$

the required condition which must be satisfied in order that the lines represented by equations (2) shall be tangents to the surface (1). This gives one relation between l, m, n, which, joined with $l^2+m^2+n^2=1$, leaves one of these quantities independent; we have, therefore, drawn through one point (x, y, z) a series of lines passing through one point and subject to one geometrical condition, and which must consequently constitute a *surface* of some kind, the nature of which we proceed to find.

(157) *To find the locus of the tangent lines which can be drawn to a surface at one point.*

The equations to any one tangent line are

$$\frac{x'-x}{l}=\frac{y'-y}{m}=\frac{z'-z}{n} \quad \ldots\ldots\ldots (8),$$

l, m, n being connected by the equation of condition

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}=0 \quad \ldots\ldots\ldots (7).$$

Now any particular tangent line is determined by the quantities l, m, n, and as the locus includes all the lines it must be

independent of l, m, n; so that if we eliminate these quantities between (7) and (8), we shall obtain a relation between x', y', z' and x, y, z, which, being true for all the lines, is the equation to their locus. The elimination is easily effected by multiplying each term of (7) by the corresponding member of (8), when we find

$$\frac{dF}{dx}(x'-x)+\frac{dF}{dy}(y'-y)+\frac{dF}{dz}(z'-z)=0 \ldots\ldots (9).$$

This being a linear equation in x', y', z' (which are the current co-ordinates of the tangent lines, and therefore also of their locus) shows that the locus of the tangent lines is in general a *plane*, which is called the tangent plane to the surface. I say that the locus is *in general* a plane, because it has been assumed in equation (7) that the differential coefficients $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$ do not all vanish at the point of contact. If this were the case both equation (7) and equation (9) would be nugatory, and it would be necessary to have recourse again to equation (5), but this exceptive case (which can occur only for isolated points and lines on surfaces) we shall treat of in another chapter. Some writers assume at once the existence of the tangent plane, but from what has preceded it is clear that this is a proposition requiring proof, and that in fact it is not always true.

(158) There is not so close analogy between tangent planes to surfaces and tangents to curves in two dimensions as the student might at first be disposed to imagine. It does not by any means always happen that the tangent plane touches the surface in one point only; it may touch it along a line, and it may cut it along one or more lines, which may even pass through the point of contact. These three kinds of tangent planes may be seen in the surface produced by the revolution of a circle round an axis in its own plane, but which does not meet the curve; such is the surface which bounds the ring of an anchor. For a plane perpendicular to the axis, which touches the surface at all, touches it in a circle; and all planes which touch the surface at points outside of this circle meet it in the point of contact alone, while those which touch the surface inside of the circle cut it in a curve. In order to determine in any given surface what is the

nature of the tangent plane, we must combine the equation to the tangent plane

$$(x'-x)\frac{dF}{dx}+(y'-y)\frac{dF}{dy}+(z'-z)\frac{dF}{dz}=0$$

with the equations

$$F(x, y, z)=0, \quad F(x', y', z')=0,$$

since, when the plane meets the surface, the point (x', y', z') is in the surface. If these lead to the conclusion

$$x'=x, \quad y'=y, \quad z'=z,$$

the tangent plane meets the surface at the point of contact only, otherwise it either touches the surface along a line or cuts it along one or two lines; the former is distinguished from the latter by two factors becoming equal.

(159) Generally the curve of intersection of a surface with its tangent plane has a double point at the point of contact.* (This includes the case of contact at one point only.) For if we have the equations $F(x', y', z')=0,$

$$\frac{dF}{dx}(x'-x)+\frac{dF}{dy}(y'-y)+\frac{dF}{dz}(z'-z)=0,$$

which are the equations to the surface, and the tangent plane at the point x, y, z, the direction of the tangent line of the curve of their intersection will be determined by combining their differentials with respect to x', y', z', or

$$\frac{dF}{dx'}dx'+\frac{dF}{dy'}dy'+\frac{dF}{dz'}dz'=0,$$

$$\frac{dF}{dx}dx'+\frac{dF}{dy}dy'+\frac{dF}{dz}dz'=0.$$

From these we may determine, in general, the ratios of $dx' : dy' : dz'$, which determine the direction of the tangent lines. But if $x'=x$, $y'=y$, $z'=z$, the two equations become identical, and consequently the ratios indeterminate: therefore to determine them we must employ in addition the equation

$$\frac{d^2F}{dx'^2}dx'^2+\frac{d^2F}{dy'^2}dy'^2+\frac{d^2F}{dz'^2}dz'^2$$
$$+2\frac{d^2F}{dy'dz'}dy'dz'+2\frac{d^2F}{dx'dz'}dx'dz'+2\frac{d^2F}{dx'dy'}dx'dy'=0.$$

* This remark is due to Mr. A. Cayley, Fellow of Trinity College.

The combination of these equations will give the ratios $dx' : dy' : dz'$ by means of a quadratic equation, implying that there are two directions of the tangent line at the point; *i.e.* that the point is double.

(160) The preceding remarks will enable us to explain the difference between the two kinds of ruled surfaces of which we spoke in Art. (136). A *ruled* surface is one which may be generated by the motion of a straight line which moves subject to certain conditions, and the two kinds are—that in which the successive generating lines do not intersect, or *skew* surfaces, as we have termed them, and that in which the successive generators do intersect, and therefore lie in the same plane. In both kinds of surfaces the generating lines must lie in the tangent plane: for if we consider the equation (5) as expanded in terms of r, the condition that r shall be a portion of the generating line implies that the value of r derived from (5) shall be indeterminate, and therefore that the coefficient of every term in (5) shall separately vanish. Hence, this includes the condition of tangency (7), and therefore the generating line is one of the tangent lines, and so lies in the tangent plane. Now the tangent plane at any point P of a generator is determined by the conditions of passing through this line and any other tangent line at the point P; in the same way the tangent plane at any other point P' in the same generator passes through that line and some other tangent line at the point P'. Hence, the two tangent planes pass through one common line (the generator PP'), and they would be coincident if the tangents at P and P' were in the same plane. But this cannot be the case in skew surfaces: for since a curve may for a small space be supposed to be coincident with its tangent, we may consider the elements of the tangent lines at P and P' to lie in the surface, so that the generator, in moving from the position PP' to the consecutive one, rests on these tangent lines; and if they were in one plane, the two positions of the generator would be also in the same plane, which is inconsistent with the definition of skew surfaces. Hence in these the tangent planes at each point in a generating line are distinct, so that in skew surfaces a tangent plane touches the

surface in one point only, but cuts it along the generating line. On the other hand, in ruled surfaces in which the consecutive positions of the generator intersect and are therefore in the same plane, the tangent planes at each point of a generator coincide, so that a tangent plane touches the surface along a generating line. From this we may easily deduce the characteristic property of these surfaces; for if G_1, G_2, G_3, G_4, &c. be consecutive positions of the generator, the surface may be considered as the limit of the planes which pass through G_1G_2, G_2G_3, G_3G_4, &c. Now we can turn each of these elemental plane areas round the line which is common to it and the succeeding one as a hinge until they both lie in the same plane—as G_1G_2 round G_2 till G_1G_2 and G_2G_3 are in the same plane, and so on in succession—until all the elemental plane areas lie in the same plane, and therefore the surface which is the limit of these planes may be in the same manner unfolded into a plane surface without introducing discontinuity; so that if we suppose the surface to consist of a thin flexible and inextensible substance, it could be unfolded into a plane without tearing or rumpling. On this account these surfaces are called *developable surfaces*: they will be more particularly treated of hereafter in the chapter on envelops.

(161) *Definition.* The normal to a surface at any point is a line perpendicular to the tangent plane at that point.

To find the equations to the normal. Let x', y', z' be the current co-ordinates of the normal, x, y, z those of the point in the surface through which it is drawn; then, since the normal is perpendicular to the tangent plane of which (9) is the equation, and as it passes through (x, y, z) its equations are

$$\frac{x'-x}{\frac{dF}{dx}} = \frac{y'-y}{\frac{dF}{dy}} = \frac{z'-z}{\frac{dF}{dz}} \ldots\ldots\ldots\ldots(10).$$

(162) If the equation to the surface be put in the form

$$z = f(x, y)$$

it is desirable to express the differentials of F in terms of those of z: in this case $F(x, y, z) = f(x, y) - z = 0.$

Therefore

$$\frac{dF}{dx}=\frac{d}{dx}f(x,y)=\frac{dz}{dx},\quad \frac{dF}{dy}=\frac{d}{dy}f(x,y)=\frac{dz}{dy},\quad \frac{dF}{dz}=-1,$$

so that the equation to the tangent plane becomes

$$z'-z=\frac{dz}{dx}(x'-x)+\frac{dz}{dy}(y'-y)\ldots\ldots\ldots(11),$$

and the equations to the normal

$$x'-x+\frac{dz}{dx}(z'-z)=0,\quad y'-y+\frac{dz}{dy}(z'-z)=0\ldots(12).$$

(163) On comparing the equation to the tangent plane (9) with the equation to the plane in Art. (39) we see that the direction-cosines of the tangent plane, and therefore of the normal, are

$$\frac{\dfrac{dF}{dx}}{\left\{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}},\quad \frac{\dfrac{dF}{dy}}{\left\{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}},$$

$$\frac{\dfrac{dF}{dz}}{\left\{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}\ldots\ldots\ldots(13).$$

The perpendicular from the origin on the tangent plane is, by Art. (48),

$$\frac{x\dfrac{dF}{dx}+y\dfrac{dF}{dy}+z\dfrac{dF}{dz}}{\left\{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}\ldots\ldots\ldots(14).$$

(164) If the equations to the surface be in the form

$$u=c,$$

where u is a homogeneous function of n dimensions, the equation to the tangent plane is much simplified. For, by a well-known property of such functions,

$$x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}=nu=nc;$$

so that the equation to the tangent plane becomes

$$x'\frac{du}{dx}+y'\frac{du}{dy}+z'\frac{du}{dz}=nc\ldots\ldots\ldots(15);$$

and the length of the perpendicular from the origin is

$$\frac{nc}{\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right\}^{\frac{1}{2}}} \dots\dots\dots\dots(16).$$

(165) *Curve of contact.* If, instead of finding the locus of all the tangent lines which touch a surface at one point, we subject them to any geometrical condition, their points of contact will trace out on the surface a line which is called the *curve of contact.*

In the case in which all the tangent lines are constrained to pass through some point not in the surface, the curve of contact is easily determined. For since the tangent line is contained in the tangent plane at any point, it is sufficient to make the latter pass through the given point; we thus find the equation to a surface which, by its intersection with the given surface, determines the curve of contact. Let the equation to the surface be

$$F(x, y, z) = 0,$$

and that to the tangent plane

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0.$$

If this pass through a point α, β, γ, we must substitute α, β, γ for x', y', z', and we have

$$(\alpha - x)\frac{dF}{dx} + (\beta - y)\frac{dF}{dy} + (\gamma - z)\frac{dF}{dz} = 0 \dots\dots(17).$$

This, considered as a function of x, y, z, represents a surface which, by its intersection with $F(x, y, z) = 0$, determines the points of contact of lines passing through α, β, γ, that is, the curve of contact.

(166) It is easy to shew that in this case the curve of contact always lies on a surface of a degree less by unity than the degree of the given surface. Let the equation to the given surface be put in the form

$$u = u_n + u_{n-1} + u_{n-2} + \&c. + u_1 = c,$$

where the different terms are homogeneous functions of the

degree indicated by the suffixes. Then the equation (17) becomes

$$(\alpha - x)\left\{\frac{du_n}{dx}+\frac{du_{n-1}}{dx}+\frac{du_{n-2}}{dx}+\&c.\right\}+(\beta-y)\left\{\frac{du_n}{dy}+\frac{du_{n-1}}{dy}+\frac{du_{n-2}}{dy}+\&c.\right\}$$
$$+(\gamma - z)\left\{\frac{du_n}{dz}+\frac{du_{n-1}}{dz}+\frac{du_{n-2}}{dz}+\&c.\right\},$$

or $$\alpha\frac{du}{dx}+\beta\frac{du}{dy}+\gamma\frac{du}{dz} = x\left\{\frac{du_n}{dx}+\frac{du_{n-1}}{dx}+\frac{du_{n-2}}{dx}+\&c.\right\}$$
$$+y\left\{\frac{du_n}{dy}+\frac{du_{n-1}}{dy}+\frac{du_{n-2}}{dy}+\&c.\right\}+z\left\{\frac{du_n}{dz}+\frac{du_{n-1}}{dz}+\frac{du_{n-2}}{dz}+\&c.\right\}.$$

But, by the well-known property of homogeneous functions,

$$x\frac{du_n}{dx}+y\frac{du_n}{dy}+z\frac{du_n}{dz}=nu_n,\quad x\frac{du_{n-1}}{dx}+y\frac{du_{n-1}}{dy}+z\frac{du_{n-1}}{dz}=(n-1)u_{n-1};$$
$$\&c. \qquad\qquad \&c.$$

therefore, observing also that

$$u_n + u_{n-1} + u_{n-2} + \&c. = c,$$

the preceding equation is reduced to

$$\alpha\frac{du}{dx}+\beta\frac{du}{dy}+\gamma\frac{du}{dz} = nc - u_{n-1} - 2u_{n-2} - 3u_{n-3} - \&c. \ldots (18).$$

And as u is of n dimensions, its differential coefficients must be each of $n - 1$ dimensions; therefore the curve of contact lies on a surface of $n - 1$ dimensions in x, y, z, or of a degree less by unity than the given surface.

Cor. Hence we see that in surfaces of the second degree the curve of contact is a plane curve.

It is evident that the curve of contact, which has just been determined, is the apparent outline of the surface to an eye placed at the point through which all the tangent lines pass; or it is the boundary between light and shade if the surface be illuminated by rays issuing from that point.

(167) When the point is removed to an infinite distance, the formulæ for the curve of contact fail, but we easily obtain more simple equations. For in that case all the tangent lines are

parallel, and therefore the direction-cosines l, m, n are constant; so that the condition which they must satisfy,

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}=0,$$

is a relation between x, y, z and constants which, combined with the equation to the surface, determines the curve of contact. It is obvious that this equation is of a degree less by unity than that of the given surface.

(168) We proceed to illustrate the preceding theory by applying it to the surfaces of the second order. The general equation to these is

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E=0\ldots(1);$$

hence, by equation (9) of Art. (157), the equation to the tangent plane at the point (x, y, z) is

$$(Ax+B'z+B''y+C)(x'-x)+(A'y+Bz+B''x+C')(y'-y)$$
$$+(A''z+By+B'x+C'')(z'-z)=0\ldots.(2),$$

x', y', z' being the current co-ordinates of the plane. On reducing this by (1), we have

$$(Ax+B'z+B''y+C)x'+(A'y+Bz+B''x+C')y'+(A''z+By+B'x+C'')z'$$
$$+Cx+C'y+C''z+E=0\ldots.(3).$$

If we apply to (1) the method of finding the centre of the surface used in Art. (78), we shall find, calling the co-ordinates of the centre α, β, γ, the conditions that the terms involving the first powers of the variables shall vanish to be

$$A\alpha+B''\beta+B'\gamma+C=0,\quad A'\beta+B\gamma+B''\alpha+C'=0,\quad A''\gamma+B\beta+B'\alpha+C''=0.$$

By means of these, eliminating C, C', C'' from (2), the equation to the tangent plane becomes

$$\{A(x-\alpha)+B'(z-\gamma)+B''(y-\beta)\}(x'-x)+\{A'(y-\beta)+B(z-\gamma)$$
$$+B''(x-\alpha)\}(y'-y)+\{A''(z-\gamma)+B(y-\beta)+B'(x-\alpha)\}(z'-z)=0\ldots(4).$$

Now if l, m, n be the direction-cosines of the diameter passing through the point (x, y, z), we have

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

which, combined with (4), give the equation

$$(Al+B'n+B''m)(x'-x)+(A'm+Bn+B''l)(y'-y)$$
$$+(A''n+Bm+B'l)(z'-z)=0\ldots.(5).$$

(169) On comparing equation (5) with the first equation of Art. (98), it will be seen that the coefficients of the variables in the two are the same, and consequently that the planes are parallel. Hence the tangent plane at any point of a surface of the second order is parallel to the plane diametral to the diameter passing through the point of contact. And from this it follows that the six planes drawn at the extremities of three conjugate diameters and parallel to their diametral planes are tangents to the surface, so that the parallelopiped formed by them circumscribes it. The volume of this parallelopiped is eight times that of the parallelopiped of which three conjugate diameters are conterminous edges, and therefore by Art. (106) is constant. The following are more particular examples.

(170) The equation to the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and therefore, by (15), the equation to the tangent plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

To determine whether this plane meets the surface in more points than one, we have to combine the preceding equations with

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1.$$

Subtracting the second equation multiplied by 2 from the sum of the first and third, we have

$$\frac{(x'-x)^2}{a^2} + \frac{(y'-y)^2}{b^2} + \frac{(z'-z)^2}{c^2} = 0.$$

This can be satisfied only by

$$x' = x, \quad y' = y, \quad z' = z,$$

and therefore the tangent plane meets the surface in one point only, which is the point of contact.

The equations to the normal at the point x, y, z are, by equation (10) of Art. (161),

$$\frac{a^2(x'-x)}{x} = \frac{b^2(y'-y)}{y} = \frac{c^2(z'-z)}{z}.$$

If p be the length of the perpendicular from the centre, on the tangent plane, we have by equation (16) of Art. (164)

$$\frac{1}{p} = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}}.$$

To find the locus of the intersection of the tangent plane with the perpendiculars on it from the centre.

The equation to the tangent plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1 \ldots\ldots\ldots\ldots (a);$$

The equations to a line through the centre, perpendicular to it, and therefore parallel to the normal, are

$$\frac{a^2x'}{x} = \frac{b^2y'}{y} = \frac{c^2z'}{z} \ldots\ldots\ldots\ldots (b).$$

At the point of intersection x', y', z' are the same in (a) and (b), while x, y, z satisfy the equation to the ellipsoid. Equation (b) may be put in the form

$$\frac{ax'}{\frac{x}{a}} = \frac{by'}{\frac{y}{b}} = \frac{cz'}{\frac{z}{c}} = (a^2x'^2 + b^2y'^2 + c^2z'^2)^{\frac{1}{2}} \ldots\ldots (c),$$

by the Theorem I. of Art (22). Multiplying each term of (a) by the corresponding member of (c), and squaring, we have

$$(x'^2 + y'^2 + z'^2)^2 = a^2x'^2 + b^2y'^2 + c^2z'^2,$$

which is the required equation. This surface is the *Surface of Elasticity* in the Wave Theory of Light.

If we call l, m, n the direction-cosines of the normal,

$$l = k\frac{x}{a^2}, \quad m = k\frac{y}{b^2}, \quad n = k\frac{z}{c^2},$$

from which, by multiplying by a, b, c, squaring and adding, we find

$$k^2 = a^2l^2 + b^2m^2 + c^2n^2,$$

so that the equation to the tangent plane may be put in the form

$$lx + my + nz = (a^2l^2 + b^2m^2 + c^2n^2)^{\frac{1}{2}}.$$

(171) The equation to the hyperboloid of one sheet is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

and therefore the equation to the tangent plane at (x, y, z) is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} - \frac{z'z}{c^2} = 1.$$

To find where this meets the surface, we combine the equations

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 1, \qquad \frac{x'x}{a^2} + \frac{y'y}{b^2} - \frac{z'z}{c^2} = 1,$$

with the equation to the surface, when we have

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) - \left(\frac{x'x}{a^2} + \frac{y'y}{b^2}\right)^2 = \left(1 + \frac{z^2}{c^2}\right)\left(1 + \frac{z'^2}{c^2}\right) - \left(1 + \frac{zz'}{c^2}\right)^2;$$

which may be reduced to

$$\left(\frac{xy' - yx'}{ab}\right)^2 = \left(\frac{z' - z}{c}\right)^2.$$

This may be split into the two linear equations

$$\frac{xy' - yx'}{ab} - \frac{z' - z}{c} = 0, \text{ and } \frac{xy' - yx'}{ab} + \frac{z' - z}{c} = 0:$$

either of which, combined with the equation to the tangent plane, gives the equations to a straight line. Consequently the tangent plane cuts the surface in two straight lines, which are in fact the generating lines passing through the point of contact.

(172) For the cone $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0,$

we find the equation to the tangent plane to be

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} - \frac{zz'}{c^2} = 0:$$

and the relation for determining the points where the tangent plane meets the surface, will be found to be

$$\left(\frac{xy' - x'y}{ab}\right)^2 = 0,$$

which, since it consists of two equal factors, shews that the tangent plane touches the surface along the line determined by the equations

$$\frac{x'}{x} = \frac{y'}{y}, \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} - \frac{zz'}{c^2} = 0,$$

that is, the generating line passing through the point of contact.

If the vertex be the point of contact, or if $x = 0$, $y = 0$, $z = 0$, the equation to the tangent plane becomes nugatory, since each term vanishes by itself. In fact there are an infinite number of tangent planes at that point: for since the tangent plane at any point touches the surface along a generating line, and all the generating lines pass through the vertex, the tangent plane at every point of the surface passes through that point. The vertex of a cone is the simplest case of the kind of singular point to which allusion is made in Art. (157).

(173) Let the equation to the hyperbolic paraboloid be in the form

$$xy = az;$$

then the equation to the tangent plane is

$$x'y + y'x = a(z' + z),$$

which meets the surface in two generators determined by its intersection with the planes

$$x' = x, \quad y' = y.$$

The equations to the normal are

$$\frac{x' - x}{y} = \frac{y' - y}{x} = -\frac{z' - z}{a}.$$

To find the locus of the intersection of the tangent plane with the perpendicular on it from the origin, we have to combine the equation to the tangent plane with

$$\frac{x'}{y} = \frac{y'}{x} = -\frac{z'}{a}.$$

Each of these ratios is by Art. (22) equal to

$$\frac{x'^2 + y'^2 + z'^2}{x'y + y'x - az'} = \frac{x'^2 + y'^2 + z'^2}{az} = \frac{x'^2 + y'^2 + z'^2}{xy},$$

in virtue of the equations to the tangent plane and to the surface: hence, we find

$$x = \frac{x'^2 + y'^2 + z'^2}{x'}, \quad y = \frac{x'^2 + y'^2 + z'^2}{y'}, \quad z = -\frac{x'^2 + y'^2 + z'^2}{z'},$$

On substituting these values of x, y, z in the equation to the surface, and dividing by $x'^2 + y'^2 + z'^2$, we find as the required equation to the locus

$$z'(x'^2 + y'^2 + z'^2) + ax'y' = 0.$$

(174) *In a central surface of the second order to find the curve of contact when the tangent lines pass all through one point.*

Let the equation to the surface be

$$Ax^2 + A'y^2 + A''z^2 = 1;$$

then by equation (15) of Art. (164) the equation to the tangent plane is

$$Axx' + A'yy' + A''zz' = 1,$$

x', y', z' being the current co-ordinates of the plane. If the co-ordinates of the fixed point be α, β, γ, the condition that the tangent plane shall pass through it gives

$$A\alpha x + A'\beta y + A''\gamma z = 1,$$

the equation to a plane which by its intersection with the surface determines the curve of contact. Since the equations to the line joining the centre of the surface with the point α, β, γ, are

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma},$$

it appears that the curve of contact is parallel to the plane, Art. 99 which is diametral to chords parallel to this line, and consequently, by Art. (121), this line passes through the centre of the curve of contact.

When the tangent lines are all parallel the equation of Art. (167) gives $$Alx + A'my + A''nz = 0,$$

the equation to a plane passing through the centre, which by its intersection with the surface determines the curve of contact. It is obvious that this is the diametral plane to chords parallel to the tangent lines.

(175) *To find the conditions that two central surfaces of the second order shall intersect everywhere at right angles.* If the surfaces cut each other at right angles, the tangent planes at every point of their line of intersection must be perpendicular. Now let the equations to the surfaces be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \ldots. (1);$$

then the equations to their tangent planes at a point (x, y, z) are

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1, \qquad \frac{xx'}{\alpha^2}+\frac{yy'}{\beta^2}+\frac{zz'}{\gamma^2}=1 \;\ldots\ldots\; (2);$$

from which we see that their direction-cosines are proportional to

$$\frac{x}{a^2},\ \frac{y}{b^2},\ \frac{z}{c^2}, \quad \frac{x}{\alpha^2},\ \frac{y}{\beta^2},\ \frac{z}{\gamma^2};$$

and since the planes are to be perpendicular, we have

$$\frac{x^2}{a^2\alpha^2}+\frac{y^2}{b^2\beta^2}+\frac{z^2}{c^2\gamma^2}=0 \;\ldots\ldots\ldots\ldots\; (3).$$

But on subtracting the second equation of (1) from the first, we have

$$x^2\left(\frac{1}{a^2}-\frac{1}{\alpha^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{\beta^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{\gamma^2}\right)=0 \;\ldots\ldots\; (4).$$

Equations (3) and (4) are two relations between x, y, z the co-ordinates of any point of intersection of the surfaces; and as they are of the same form they must be identical, with the exception of a factor. Multiplying then (3) by λ, and equating the coefficients of the variables in the two equations, we have

$$a^2-\alpha^2=b^2-\beta^2=c^2-\gamma^2=\lambda;$$

whence $a^2-b^2=\alpha^2-\beta^2$, $a^2-c^2=\alpha^2-\gamma^2$, $b^2-c^2=\beta^2-\gamma^2$.

These latter equations show that the principal sections of the surfaces have the same foci: such surfaces are called *confocal*, and possess many remarkable properties. Since equation (3) cannot be true generally, unless one at least of the terms be negative, it appears that one of the surfaces must be a hyperboloid.

CHAPTER IX.

OF TANGENTS TO CURVES, NORMAL AND OSCULATING PLANES.

(176) A curve in general is determined by the intersection of two surfaces, and at every point of the line of intersection two tangent planes can be drawn, one to each surface; the intersection of these two planes is a tangent line to the curve. To find its equations let
$$u = 0, \quad v = 0$$
be the equations to the two surfaces; the equations to their tangent planes are

$$(x' - x)\frac{du}{dx} + (y' - y)\frac{du}{dy} + (z' - z)\frac{du}{dz} = 0,$$

$$(x' - x)\frac{dv}{dx} + (y' - y)\frac{dv}{dy} + (z' - z)\frac{dv}{dz} = 0.$$

As these are drawn at the same point, the co-ordinates x, y, z are the same in both; and when they intersect, x', y', z' are the same. Hence, eliminating the quantities $z' - z$, $y' - y$, $x' - x$ in succession, we find

$$\frac{x' - x}{\frac{du}{dy}\frac{dv}{dz} - \frac{du}{dz}\frac{dv}{dy}} = \frac{y' - y}{\frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz}} = \frac{z' - z}{\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}} \dots (1),$$

as the equations to the line which is a tangent at the point (x, y, z) to the curve determined by the intersection of the surfaces
$$u = 0, \quad v = 0.$$
From the equations

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz = 0,$$

$$\frac{dv}{dx}dx + \frac{dv}{dy}dy + \frac{dv}{dz}dz = 0.$$

we find

$$\frac{dx}{\frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dy}}=\frac{dy}{\frac{du}{dz}\frac{dv}{dx}-\frac{du}{dx}\frac{dv}{dz}}=\frac{dz}{\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}} \dots (2),$$

so that the equations to the tangent line may be put under the form

$$\frac{x'-x}{dx}=\frac{y'-y}{dy}=\frac{z'-z}{dz} \dots\dots\dots\dots (3),$$

or
$$x'-x=\frac{dx}{dz}(z'-z), \quad y'-y=\frac{dy}{dz}(z'-z) \dots\dots (4),$$

a form which is convenient when two of the variables are given as functions of the third.

(177) If one of the equations, as $u = 0$, do not involve one of the variables as z, it represents the projection of the curve on the plane of xy, and the equation

$$(x'-x)\frac{du}{dx}+(y'-y)\frac{du}{dy}=0$$

represents both the projection of the tangent on the plane of (x, y) and the tangent at the corresponding point of the projection of the curve on that plane. Hence, generally, the projection on any plane of the tangent to a curve coincides with the tangent at the corresponding point of the projection of the curve on that plane.

(178) *Normal plane.* A normal to a curve being defined as a straight line drawn perpendicular to a tangent through the point of contact, it is clear that there are an infinite number of such lines all lying in a plane perpendicular to the tangent line, which is called the normal plane. The equation to this plane is evidently

$$\left(\frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dy}\right)(x'-x)+\left(\frac{du}{dz}\frac{dv}{dx}-\frac{du}{dx}\frac{dv}{dz}\right)(y'-y)$$
$$+\left(\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}\right)(z'-z)=0 \dots\dots (5);$$

or
$$(x'-x)\,dx+(y'-y)\,dy+(z'-z)\,dz=0 \dots\dots (6).$$

The differentials in this equation are to be got rid of by means

of the equations to the curve, from which we can find the ratios $dx : dy : dz$.

(179) If we put

$$\Delta^2 = \left(\frac{du}{dy}\frac{dv}{dz} - \frac{du}{dz}\frac{dv}{dy}\right)^2 + \left(\frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz}\right)^2 + \left(\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right)^2,$$

the direction-cosines of the tangent and of the normal plane are, from equations (1) and (5),

$$\frac{1}{\Delta}\left(\frac{du}{dy}\frac{dv}{dz} - \frac{du}{dz}\frac{dv}{dy}\right), \quad \frac{1}{\Delta}\left(\frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz}\right), \quad \frac{1}{\Delta}\left(\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right) \quad \ldots\ldots(7),$$

or, from equations (3) and (6),

$$\frac{dx}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}}, \quad \frac{dy}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}}, \quad \frac{dz}{(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}} \ldots(8).$$

But by the direct application of geometrical conceptions, these may be expressed in simpler forms. A curve, whether plane or of double curvature, may be considered as the limit of a polygon, the sides of which are diminished in magnitude while they are increased in number indefinitely. Now if Δs be the length of a side of the polygon, the difference of the co-ordinates of the extremities of which are Δx, Δy, Δz, the direction-cosines of the side of the polygon are

$$\frac{\Delta x}{\Delta s}, \quad \frac{\Delta y}{\Delta s}, \quad \frac{\Delta z}{\Delta s}.$$

But as the sides of the polygon are diminished in magnitude while their number is increased indefinitely, the side coincides ultimately with the tangent to the curve, and the limits of these ratios are consequently the direction-cosines of the tangent; and as the length of the side is ultimately the increment of the arc s of the curve, the limits of these ratios or the direction-cosines of the tangent are, in the language of the differential calculus, expressed by

$$\frac{dx}{ds}, \quad \frac{dy}{ds}, \quad \frac{dz}{ds}.$$

Hence
$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1;$$

or
$$ds^2 = dx^2 + dy^2 + dz^2.$$

(180) *Osculating plane.* If we consider a curve as the limit

of a polygon, the sides of the latter are not necessarily all in one plane, but any two conterminous sides must be in the same plane, since they intersect. This plane passes through three of the angular points of the polygon; and as the polygon approaches to the curve as its limit, this plane tends to assume ultimately a determinate position, in which it is called the *osculating plane.* To find its equation we must express the condition that it passes through three consecutive points of the curve; that is, through three points of which the co-ordinates are

$$x, \quad y, \quad z,$$
$$x + dx, \quad y + dy, \quad z + dz,$$
$$x + 2dx + d^2x, \quad y + 2dy + d^2y, \quad z + 2dz + d^2z.$$

Now, if in any equation $F(x, y, z) = 0$ we substitute these values of the co-ordinates, it becomes in succession,

$$F(x, y, z) + dF(x, y, z) = 0,$$
$$F(x, y, z) + 2dF(x, y, z) + d^2F(x, y, z) = 0;$$

and when these are taken simultaneously they are equivalent to

$$F(x, y, z) = 0, \quad dF(x, y, z) = 0, \quad d^2F(x, y, z) = 0.$$

The equation to a plane passing through a point (x, y, z) may be assumed to be

$$A(x' - x) + B(y' - y) + C(z' - z) = 0 \ldots\ldots (1);$$

and if it be the osculating plane, that is, if it pass through three consecutive points of the curve, we must have, by what has just been said,

$$Adx + Bdy + Cdz = 0 \ldots\ldots\ldots\ldots (2),$$
$$Ad^2x + Bd^2y + Cd^2z = 0 \ldots\ldots\ldots\ldots (3),$$

Eliminating A, B, C in turn between (2) and (3), we have

$$\frac{A}{dyd^2z - dzd^2y} = \frac{B}{dzd^2x - dxd^2z} = \frac{C}{dxd^2y - dyd^2x} \ldots (4);$$

so that by eliminating A, B, C between (1) and (4), the equation to the osculating plane is

$$(dyd^2z - dzd^2y)(x' - x) + (dzd^2x - dxd^2z)(y' - y)$$
$$+ (dxd^2y - dyd^2x)(z' - z) = 0 \ldots (5).$$

It is obvious that this is perpendicular to the normal plane, and therefore passes through the tangent.

In applying equation (5), if we keep it in the symmetrical form, we must consider x, y, z as functions of some other

variable; but if we consider any one of them to be independent, of which the other two are functions given by the equations to the curve, the formula necessarily becomes unsymmetrical. Thus, if x be taken as the independent variable, so that $d^2x = 0$, equation (5) becomes

$$\left(\frac{dy}{dx}\frac{d^2z}{dx^2} - \frac{dz}{dx}\frac{d^2y}{dx^2}\right)(x' - x) - \frac{d^2z}{dx^2}(y' - y) + \frac{d^2y}{dx^2}(z' - z) = 0 \;\ldots(6).$$

The following are applications of the preceding formulæ to particular examples.

(181) The equations of the equable spherical spiral are, by Art. (152), $x^2 + y^2 + z^2 = 4r^2, \quad x^2 + y^2 - 2rx = 0,$

or $\qquad z^2 + 2rx = 4r^2, \quad x^2 + y^2 - 2rx = 0.$

From these we find

$$\frac{dy}{dx} = \frac{r - x}{y}, \quad \frac{dz}{dx} = -\frac{r}{z},$$

so that the equations to the tangent become, after obvious reductions, $\quad yy' + (x - r)\,x' = rx, \quad zz' + rx' = 4r^2 - rx;$

and the equation to the normal plane is

$$(x' - x) + (y' - y)\frac{r - x}{y} - (z' - z)\frac{r}{z} = 0.$$

This, after reduction, may be put in the form

$$\left(\frac{x'}{x} - \frac{y'}{y}\right) = \frac{r}{x}\left(\frac{z'}{z} - \frac{y'}{y}\right),$$

in which we see that it is satisfied by the equations

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z},$$

and therefore it contains the radius of the sphere drawn to the point (x, y, z).

The equation to the osculating plane is, taking x as the independent variable,

$$x'\{xy^2 - r(y^2 - z^2)\} + y'y^3 + z'z^3 = 4r^2z^2 + rx(y^2 - z^2).$$

(182) The equations to the spherical ellipse are (Art. 153)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad x^2 + y^2 + z^2 = r^2,$$

or $$\frac{x^2-f^2}{a^2(c^2-b^2)}=\frac{y^2-g^2}{b^2(a^2-c^2)}=\frac{z^2-h^2}{c^2(b^2-a^2)}.$$

The equations to the tangent are

$$\frac{x(x'-x)}{a^2(c^2-b^2)}=\frac{y(y'-y)}{b^2(a^2-c^2)}=\frac{z(z'-z)}{c^2(b^2-a^2)},$$

or $$\frac{xx'-f^2}{a^2(c^2-b^2)}=\frac{yy'-g^2}{b^2(a^2-c^2)}=\frac{zz'-h^2}{c^2(b^2-a^2)}.$$

The equation to the normal plane is

$$a^2(c^2-b^2)\frac{x'}{x}+b^2(a^2-c^2)\frac{y}{y'}+c^2(b^2-a^2)\frac{z'}{z}=0,$$

and therefore it passes always through the centre of the sphere.

The equation to the osculating plane is somewhat complex, but if we put

$$a^2(c^2-b^2)=A,\quad b^2(a^2-c^2)=B,\quad c^2(b^2-a^2)=C,$$

and $$Bz^2-Cy^2=Bh^2-Cg^2=L,$$
$$Cx^2-Az^2=Cf^2-Ah^2=M,$$
$$Ay^2-Bx^2=Ag^2-Bf^2=N,$$

it takes the very symmetrical form

$$\frac{L}{A}x^3(x'-x)+\frac{M}{B}y^3(y'-y)+\frac{N}{C}z^3(z'-z)=0.$$

(183) The equations to the helix being in the form

$$x=a\cos\frac{z}{h},\quad y=a\sin\frac{z}{h},\quad x^2+y^2=a^2,$$

the equations to the tangent are

$$\frac{x'-x}{-y}=\frac{y'-y}{x}=\frac{z'-z}{h}.$$

If θ be the angle which this line makes with the axis of z, we find $\tan\theta=\frac{a}{h}$: θ is therefore constant, and the tangent is inclined to the plane of (x, y) always at the same angle. The tangent meets the plane of (x, y) in points forming a curve, the equation to which is easily found. For, making $z'=0$ in the equations to the tangent, we have

$$h(x'-x)-yz=0,\quad h(y'-y)+xz=0,$$

between which and the equations to the curve we have to eliminate x, y, z. The last equations may be put in the form

$$hx' = hx + yz, \quad hy' = hy - xz;$$

whence, by squaring and adding,

$$h^2(x'^2 + y'^2) = h^2a^2 + a^2z^2,$$

and therefore
$$z^2 = \frac{h^2}{a^2}(x'^2 + y'^2 - a^2),$$

$$x = a \cos \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a}, \quad y = a \sin \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a}.$$

But multiplying the equations by x and y and adding, we have

$$xx' + yy' = a^2;$$

substituting in this for x and y, their preceding values, we have

$$x' \cos \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a} + y' \sin \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a} = a;$$

which is the equation to the involute of the circle.

The equation to the normal plane is

$$xy' - yx' + h(z' - z) = 0.$$

The equation to the osculating plane will be most readily found by making z the independent variable, and therefore $d^2z = 0$; this gives us
$$h(x'y - y'x) + a^2(z' - z) = 0.$$

In both of these equations, if we make $x' = 0$, $y' = 0$, we find $z' = z$, that is, both planes cut the axis of z at the same point, which is the corresponding co-ordinate of the point in the curve: their line of intersection is therefore parallel to the plane of (x, y). It is easy to see that both planes are inclined at a constant angle to the plane of (x, y), the direction-cosine of the normal plane being

$$\frac{h}{(x^2 + y^2 + h^2)^{\frac{1}{2}}} = \frac{h}{(a^2 + h^2)^{\frac{1}{2}}},$$

and that of the osculating plane being

$$\frac{a^2}{\{a^4 + h^2(x^2 + y^2)\}^{\frac{1}{2}}} = \frac{a}{(a^2 + h^2)^{\frac{1}{2}}},$$

the complement of the former, as is otherwise obvious.

Line of Greatest Slope.

(184) The line of greatest slope on a surface, starting from any point, is such that its tangent always makes a greater angle with a given plane than any other tangent line to the surface drawn through the same point. Since all the tangent lines at any point of a surface lie in the tangent plane, that one which is perpendicular to the intersection of the tangent plane with the given plane, makes the greatest angle with the plane. Hence the line of greatest slope on any surface has its tangent at every point perpendicular to the intersection of the tangent plane and the given plane. Take the plane of (x, y) as the given plane, and let the equation to the surface be

$$F(x, y, z) = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

so that the equation to the tangent plane at the point (x, y, z) is

$$\frac{dF}{dx}(x' - x) + \frac{dF}{dy}(y' - y) + \frac{dF}{dz}(z' - z) = 0\,;$$

the intersection of which with the plane of (x, y) is given by the equations

$$z' = 0, \quad \frac{dF}{dx}x' + \frac{dF}{dy}y' = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} \quad \ldots \quad (2).$$

The equations to the tangent to the required curve, at the point (x, y, z), are in the form

See Hymers A. 176.

$$\frac{x' - x}{dx} = \frac{y' - y}{dy} = \frac{z' - z}{dz} \quad \ldots\ldots\ldots \quad (3);$$

and if the lines (2) and (3) are perpendicular, we have

$p\,dy - q\,dx$

$$\frac{dF}{dx}dy - \frac{dF}{dy}dx = 0 \quad \ldots\ldots\ldots\ldots \quad (4).$$

This differential equation, which expresses the general property of the line of greatest slope, combined with the equation (1), will serve to determine it; and if between (1) and (4) we eliminate z, we shall obtain a differential equation between x and y, which on integration gives the projection of the line of greatest slope on the plane of (x, y).

(185) Let the given surface be the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the given plane, as before, the plane of (x, y); then equation (4) becomes

$$\frac{x}{a^2}\,dy - \frac{y}{b^2}\,dx = 0,$$

or

$$a^2\,\frac{dx}{x} - b^2\,\frac{dy}{y} = 0;$$

in which the variable z does not appear. The integral is $a^2 \log x = b^2 \log y + C$

or. $$x^{a^2} = Cy^{b^2}.$$

The constant C is to be determined by the condition of the line of greatest slope passing through any given point: if the co-ordinates of this be α, β, the preceding equation becomes

$$\left(\frac{x}{\alpha}\right)^{a^2} = \left(\frac{y}{\beta}\right)^{b^2}.$$

The intersection of this cylinder with the ellipsoid will give in general a curve of double curvature: but if the point (α, β) lie in one of the principal planes of (y, z) or (x, z) the line of greatest slope is plane; for if it lie in the plane of (y, z), $\alpha = 0$, and therefore $x = 0$, showing that the section of the ellipsoid by the plane of (y, z), is the line of greatest slope.

(186) Let the given surface be that represented by the general equation

$$z = \phi\left(\frac{y}{x}\right),$$

the nature of which will be explained in the following chapter. In this case equation (4) becomes

$$xdx + ydy = 0,$$

the integral of which is

$$x^2 + y^2 = c^2;$$

so that the line of greatest slope in these surfaces is projected on the plane of (x, y) in a circle, the centre of which is at the origin; that is, the line is determined by the intersection of the surface with a right circular cylinder, of which the axis is the axis of z.

CHAPTER X.

OF THE GENERATION OF SURFACES BY THE MOTION OF CURVES.

(187) In investigating the equation to the plane (Art. 33), and those to the hyperboloid of one sheet, and hyperbolic paraboloid (Arts. 138 and 141), we considered these surfaces as traced out by the motion of a straight line constrained to move in a certain manner. An extension of this method gives us the means of conveniently classifying and discussing surfaces of which the equations are of a degree higher than the second. The general theory of the process for finding the equations to surfaces defined as generated by the motion of a curve may be explained in the following manner.

(188) A line is represented by two equations to surfaces involving x, y, z and constants; we may suppose one of the constants in each equation to be arbitrary, and to admit of an indefinite number of values, corresponding to which the surface assumes different forms or positions; for the form and position of a surface depend on the values assigned to the constants in its equation. If we assume the equations to be solved with respect to these arbitrary constants, or parameters as they are styled, they are in the form

$$f(x, y, z) = c, \quad f_1(x, y, z) = c_1 \ldots\ldots\ldots (1),$$

or, as we may write them,

$$u = c, \quad v = c_1 \ldots\ldots\ldots\ldots\ldots (1').$$

Now, on assigning different values to c and c_1, the line determined by these two equations will change in form or in position, or in both, in consequence of the change in the surfaces. If c and c_1 be independent, the line may be made to occupy all points within the space for which the equations (1′) are satisfied

by possible values of the variables, so that it will trace out a solid locus. But if we assume a relation to exist between c and c_1, which may be expressed by the equation

$$\psi(c, c_1) = 0, \quad \text{or } c_1 = \phi(c) \ldots\ldots\ldots(2);$$

then for each value of c the curve will assume only one determinate form or position, and in passing through all its forms and positions it will trace out a surface. To determine the equation to this surface it is clear that we must obtain a relation between x, y, and z, independent of the quantity c, which determines by its successive values the successive forms and positions of the curve. This is easily done by eliminating c between the equations

$$u = c, \quad v = \phi(c) \ldots\ldots\ldots\ldots\ldots(3),$$

the result of which is evidently

$$v = \phi(u) \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

(189) Hence, when the parameters in the equation to the generator are given explicitly, it is very easy to find the equation to the surface, as when the relation between the parameters is given, the elimination is at once effected. Thus, for example, in finding the equation to the plane, since we suppose the generator to move parallel to itself, the quantities a and b of the equations

$$x = az + p, \quad y = bz + q,$$

of Art. (26) are constant, and p and q are the variable parameters, so that if they be written on one side of the equations, these then become $x - az = p, \quad y - bz = q.$

Now the relation between p and q is to be found by the condition that the generator shall pass always through the director, of which the equations may be written

$$x = lz + h, \quad y = mz + k,$$

the condition for which is, by equation (5) of Art. (30),

$$\frac{p-h}{a-l} = \frac{q-k}{b-m},$$

and this equation is the form of (2) appropriate to the present case. Hence, eliminating p and q by means of the equations

$$x - az = p, \quad y - bz = q,$$

we find

$$\frac{x - az - h}{a - l} = \frac{y - bz - k}{b - m},$$

as the equation to the plane, which is evidently of the first degree in x, y, z.

(190) In what precedes we have assumed that there is only one variable parameter involved in the two equations to the generator, but the subject may be considered more generally. For we may suppose the equations to the generator to contain n parameters $c_1, c_2 \ldots c_n$, so that they may be written

$$f(x, y, z, c_1, c_2 \ldots c_n) = 0, \quad f_1(x, y, z, c_1, c_2 \ldots c_n) = 0.$$

Then in order that the motion of the generator may be completely regulated, so that it shall trace out a surface, the n parameters must be connected by $n-1$ relations, so as to leave one independent quantity only. If these relations be

$$\psi_1(c_1 \ldots c_n) = 0, \quad \psi_2(c_1 \ldots c_n) = 0 \ldots \psi_{n-1}(c_1 \ldots c_n) = 0,$$

the equation to the surface will be found by eliminating the n parameters between these $n-1$ equations and the two equations of the generator, making $n+1$ in all.

(191) It usually happens in practice that the relation between the constants is not given directly, but is to be deduced from the geometrical condition that the generator shall pass always through some given director-line, as was the case in the preceding example. It is easy to see that such a geometrical condition always corresponds to one relation between the parameters. For if the equations to the director be

$$F(x, y, z) = 0, \quad F_1(x, y, z) = 0,$$

in order to express that the generator passes constantly through it, we must make x, y, z the same in these equations and in the equations to the generator

$$f(x, y, z, c_1 \ldots c_n) = 0, \quad f_1(x, y, z, c_1 \ldots c_n) = 0.$$

Between these four equations we may eliminate the three quantities x, y, z, so that we shall obtain a result involving $c_1, c_2 \ldots c_n$ only, and which may be written as

$$\psi(c_1, c_2 \ldots c_n) = 0,$$

which is one relation between the parameters. A similar result may be obtained for every director on which the generator is constrained to rest, and consequently a curve which contains n constants in its equations, may be made to rest on $n-1$ directors.

(192) When there is only one director we may obtain the result of the final elimination without previously finding the relation between the parameters which we have denoted by the equation $c_1 = \phi(c)$. Let the equations to the generator be

$$u = c, \quad v = \phi(c),$$

u and v being functions of x, y, z; and the equations to the director $$F(x, y, z) = 0, \quad F_1(x, y, z) = 0.$$

Then if x', y', z' be the current co-ordinates of the generator, and u', v' the values of u and v, when x', y', z' are substituted for x, y, z, the equations to the generator may be written as

$$u' = u, \quad v' = v;$$

and if we eliminate x, y, z between these equations and the equations of the director, we shall obtain a result in x', y', z' independent of x, y, z, and which is therefore true for all points of the generator in all its positions, and consequently must be the equation of the required locus.

(193) Sometimes instead of assigning a director through which the generator is to pass, the geometrical condition to which it is subject is, that the locus shall circumscribe a given surface, but this condition may be at once reduced to the preceding. For let $$F(x, y, z) = 0$$

be the equation to the surface which is to be circumscribed, then the direction-cosines of the normal are proportional to $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$. On the other hand, the direction-cosines of the tangent to the curve determined by the equations

$$u = c, \quad v = c_1,$$

are proportional to

$$\frac{du}{dy}\frac{dv}{dz} - \frac{du}{dz}\frac{dv}{dy} = P, \quad \frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz} = Q,$$

$$\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx} = R.$$

But since the surface described by the generator is to circumscribe the surface $F(x, y, z) = 0$, the generator must touch it in

every position, and therefore be perpendicular to the normal. Consequently we have the condition

$$P\frac{dF}{dx} + Q\frac{dF}{dy} + R\frac{dF}{dz} = 0.$$

This is a relation between x, y, z which holds for all points in which the surfaces touch each other, and may be considered as the equation to a surface passing through the curve of contact. The generator therefore may be considered as constrained to pass through a director determined by the equations

$$F(x, y, z) = 0, \quad P\frac{dF}{dx} + Q\frac{dF}{dy} + R\frac{dF}{dz} = 0.$$

(194) It is to be observed that surfaces generated in the manner we have been considering may differ from each other in two ways—either by having different generators or different directors. Those surfaces which admit of the same generator, that is, for which the forms of the functions f and f_1, or u and v in Art. (188), are the same, are said to belong to the same *family*, while the individual surfaces are distinguished by differences in the forms of the functions ψ_1, ψ_2, &c., that is, in the nature of the directors on which the generator rests. In the examples with which we shall illustrate the theory we shall consider only two families of surfaces—those which have a straight line, and those which have a circle as generators. The former, or *ruled surfaces*, as we have previously termed them, are conveniently divided into the two classes of skew and developable surfaces, the nature of the distinction between which was explained in Arts. (136) and (160). Since the equations to a straight line contain four independent constants which may be considered as parameters, it appears from Art. (191) that no ruled surface can have more than three directors, as is the case in the hyperboloid of one sheet where they are straight lines.

(195) It is only, however, in skew surfaces that there can be so many as three directors; for since in developable surfaces the successive generators intersect, the condition that this should happen gives one relation between the parameters. Thus, if we

suppose x, y, z to be the current co-ordinates of one director, x_1, y_1, z_1 of the other, their equations being

$$F(x, y, z) = 0, \qquad F_1(x, y, z) = 0 \ldots\ldots (1),$$

$$F_2(x_1, y_1, z_1) = 0, \qquad F_3(x_1, y_1, z_1) = 0 \ldots\ldots (2),$$

the equations to the generator in any position are

$$\frac{x' - x}{x - x_1} = \frac{y' - y}{y - y_1} = \frac{z' - z}{z - z_1} \ldots\ldots\ldots\ldots (3),$$

x', y', z' being the current co-ordinates of the generator. Now, since two successive generators intersect, the points x, y, z, x_1, y_1, z_1 must be so taken that the tangents to the curves at those points are in the same plane, see Art. (160); and the equations of the tangents to (1) and (2) being

$$\frac{x' - x}{dx} = \frac{y' - y}{dy} = \frac{z' - z}{dz} \ldots\ldots\ldots\ldots (4),$$

$$\frac{x' - x_1}{dx_1} = \frac{y' - y_1}{dy_1} = \frac{z' - z_1}{dz_1} \ldots\ldots\ldots\ldots (5);$$

the condition for these lying in the same plane is, by Art. (30),

$$(dydz_1 - dy_1dz)(x - x_1) + (dzdx_1 - dxdz_1)(y - y_1) + (dxdy_1 - dx_1dy)(z - z_1) = 0 \ldots\ldots\ldots (6).$$

The differentials in this equation may be eliminated by means of the differentials of (1) and (2), and then (6) is a relation between the quantities x, y, &c., which, combined with the six equations (1), (2) and (3), gives seven equations, between which the six quantities x, y, &c., may be eliminated, and a relation obtained between x', y', z', which is the equation to the developable surface.

(196) Since the successive generators of a developable surface intersect, they will by their intersection determine a curve to which they are all tangents. This curve has been called by French writers the "Arête de rebroussement" of the developable surface, and the term has been translated into the English phrase "edge of regression": perhaps, however, the name "cuspidal edge" expresses better the meaning of the French words. This curve is a remarkable line on the surface, as it is a prominent edge which offers a salient angle in all plane sections

except those which pass through a generator. It appears from the nature of the curve that the surface falls away from it in two sheets, so that it is an extreme boundary to the surface: it is evidently a curve of double curvature, since if it were a plane curve its tangents would lie all in one plane, and the developable surface would then be reduced to a plane. The student may perhaps obtain a more distinct idea of the nature of this line by the inspection of fig. (27).

(197) Every developable surface has a cuspidal edge peculiar to itself; in the case of cones it is reduced to a point, and in cylinders this point is removed to an infinite distance. Hence, when the equations to any curve of double curvature are given, we may find the equation to the developable surface of which it is the edge; for if its equations be

$$F(x, y, z) = 0, \quad F_1(x, y, z) = 0,$$

and those to its tangent at a point x, y, z,

$$\frac{x' - x}{dx} = \frac{y' - y}{dy} = \frac{z' - z}{dz},$$

we may between these four equations eliminate x, y, z, and obtain a relation between x', y', z', which is the equation to the developable surface. The method of finding the equations to the edge when that to the surface is given, will be found in the Chapter on Singular Points and Lines in Surfaces.

There is a third mode of considering the generation of developable surfaces, which we shall explain in the next chapter.

Cylindrical Surfaces.

(198) Cylindrical surfaces are those which are generated by the motion of a straight line which always remains parallel to a given position. To find the general equation to such surfaces, let the equations to the generator be

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} \;\ldots\ldots\ldots\ldots\; (1),$$

then l, m, n, being the direction-cosines, are constant, while x, y, z vary from one position of the generator to another. The preceding equations may be written in the form

$$ly' - mx' = ly - mx, \quad lz' - nx' = lz - nx\ldots\ldots(2),$$

in each of which the second side may be looked on as a single variable parameter, so that they are given explicitly as c and c_1 in Art. (188). But in order that the generator may move so as to trace out a surface, some relation between these parameters must exist; and we may express this by writing the equation

$$lz - nx = \phi\,(ly - mx) \;\ldots\ldots\ldots\ldots\; (3).$$

Eliminating α, β, γ between (2) and (3), we have

$$lz' - nx' = \phi\,(ly' - mx') \;\ldots\ldots\ldots\ldots\; (4)$$

as the general equation to cylindrical surfaces, the individual surface being determined by the nature of the function ϕ.

(199) If we assume that the generator is determined by the more general equations

$$lx + my + nz = \delta, \quad l_1x + m_1y + n_1z = \delta_1 \;\ldots\ldots\; (5),$$

we easily find the general equation to cylindrical surfaces to be

$$l_1x + m_1y + n_1z = \phi\,(lx + my + nz) \;\ldots\ldots\ldots\ldots\; (6),$$

so that if we have given an equation in which one linear function of x, y, z is made equal to any function of another linear function of these variables, that equation represents a cylindrical surface.

(200) To find the equation to a cylinder of which the director is a plane curve determined by the equations

$$\lambda x + \mu y + \nu z = \delta, \quad F(x, y, z) = 0 \;\ldots\ldots\; (7),$$

let the equations to the generator be

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} \;\ldots\ldots\ldots\ldots\; (8);$$

between these four equations we have to eliminate x, y, and z. Each of the ratios (8) is equal to

$$\frac{\lambda\,(x' - x) + \mu\,(y' - y) + \nu\,(z' - z)}{l\lambda + m\mu + n\nu} = \frac{\lambda x' + \mu y' + \nu z' - \delta}{l\lambda + m\mu + n\nu},$$

in virtue of the first of equations (7). Hence if we put

$$l\lambda + m\mu + n\nu = k,$$

we find
$$x = x' - \frac{l}{k}\,(\lambda x' + \mu y' + \nu z' - \delta),$$

$$y = y' - \frac{m}{k}\,(\lambda x' + \mu y' + \nu z' - \delta),$$

$$z = z' - \frac{n}{k}\,(\lambda x' + \mu y' + \nu z' - \delta);$$

and these values being substituted in the equation $F(x, y, z) = 0$, give a relation between x', y', z' and constants, which is the equation to the cylinder.

(201) Let the director be the ellipse

$$z = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this case $\lambda = 0$, $\mu = 0$, $\nu = 1$, $\delta = 0$, $k = n$, whence

$$x = \frac{nx' - lz'}{n}, \quad y = \frac{ny' - mz'}{n},$$

so that the equation to the cylinder is

$$\frac{(nx' - lz')^2}{a^2} + \frac{(ny' - mz')^2}{b^2} = n^2 \ldots\ldots\ldots\ldots (9).$$

(202) Let the cylinder circumscribe the ellipsoid

$$Ax^2 + A'y^2 + A''z^2 = 1 \ \ldots\ldots\ldots (10);$$

then by Art (167) the director is determined by equation (10) combined with

$$Alx + A'my + A''nz = 0 \ \ldots\ldots\ldots (11),$$

the equations to the generator being

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} = r \ldots\ldots\ldots (12).$$

But on combining equations (11) and (12), we have in virtue of (10)

$$Axx' + A'yy' + A''zz' = 1 \ \ldots\ldots\ldots (13),$$

and we may use (11) and (13) for the elimination of x, y, z instead of (10) and (11). From (12) we have

$$x = x' - lr, \quad y = y' - mr, \quad z = z' - nr;$$

which substituted in (11) and (12) gives

$$Alx' + A'my' + A''nz' - (Al^2 + A'm^2 + A''n^2)\, r = 0 \ldots (14),$$

$$Ax'^2 + A'y'^2 + A''z'^2 - 1 - (Alx' + A'my' + A''nz')\, r = 0 \ldots (15).$$

Eliminating r between (14) and (15), we obtain

$$(Al^2+A'm^2+A''n^2)(Ax'^2+A'y'^2+A''z'^2-1) = (Alx'+A'my'+A''nz')^2 \ldots (16),$$

as the equation to the circumscribing cylinder. If ρ be the semidiameter of the ellipsoid which is parallel to the generators of the cylinder, its direction-cosines are l, m, n, and hence from (10) we find

$$Al^2 + A'm^2 + A''n^2 = \frac{1}{\rho^2},$$

which, substituted in (16), makes the equation to the cylinder

$$Ax'^2 + A'y'^2 + A''z'^2 - 1 = \rho^2 (Alx' + A'my' + A''nz')^2:$$

in this may be recognized the form (5) of (Art. 149) for the equation to a surface of the second order which circumscribes another surface of the same order.

Conical Surfaces.

(203) Conical surfaces are generated by the motion of a straight line which passes constantly through a fixed point. To find their general equation, let a, b, c be the co-ordinates of the fixed point; then the equations to the generator may be written as

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \dots\dots\dots\dots \quad (1),$$

whence
$$\frac{x-a}{z-c} = \frac{l}{n}, \quad \frac{y-b}{z-c} = \frac{m}{n} \quad \dots\dots\dots\dots \quad (2).$$

Now a, b, c are constant, while l, m, n are the variable parameters, since they vary with the position of the generator: hence in equations (2) the ratios $\frac{l}{n}$, $\frac{m}{n}$ may be considered as parameters given explicitly, and in order that the generator may in its motion trace out a surface, the one of these must be some function of the other, which is expressed by writing

$$\frac{l}{n} = \phi\left(\frac{m}{n}\right) \quad \dots\dots\dots\dots \quad (3),$$

whence eliminating l, m, n by means of (2), we have

$$\frac{x-a}{z-c} = \phi\left(\frac{y-b}{z-c}\right) \dots\dots\dots\dots \quad (4)$$

as the general equation to conical surfaces.

If the fixed point be taken as origin $a = 0$, $b = 0$, $c = 0$, and the equation (4) becomes

$$\frac{x}{z} = \phi\left(\frac{y}{z}\right) \quad \dots\dots\dots\dots \quad (5).$$

This may be written as

$$\psi\left(\frac{x}{z}, \frac{y}{z}\right) = 0,$$

in which form we see that it is equivalent to saying that it is a

homogeneous function of x, y, z equated to zero: so that we have thus extended to all conical surfaces the remark made in Art. (81) on cones of the second degree.

(204) Let the cone be that of which the director is the plane curve determined by the equations

$$lx + my + nz = \delta, \quad F(x, y, z) = 0 \ldots\ldots (6).$$

If the co-ordinates of the vertex be a, b, c, the equations to the generator may be written as

$$\frac{x' - a}{x - a} = \frac{y' - b}{y - b} = \frac{z' - c}{z - c} \quad \ldots\ldots\ldots (7),$$

since it passes through the point (a, b, c), and also through a point x, y, z of the director. Each of these ratios is equal to

$$\frac{l(x' - a) + m(y' - b) + n(z' - c)}{l(x - a) + m(y - b) + n(z - c)} = \frac{l(x' - a) + m(y' - b) + n(z' - c)}{\delta - (la + mb + nc)},$$

in virtue of the first of equations (6). Hence if we put

$la + mb + nc = d$, we find

$$x = a - \frac{(d - \delta)(x' - a)}{lx' + my' + nz' - d},$$

$$y = b - \frac{(d - \delta)(y' - b)}{lx' + my' + nz' - d},$$

$$z = c - \frac{(d - \delta)(z' - c)}{lx' + my' + nz' - d},$$

which values substituted in $F(x, y, z) = 0$, give a relation between x', y', z' which is the required equation. If the director lie in the plane of (x, y), so that $l = 0$, $m = 0$, $n = 1$, $\delta = 0$, $d = c$, the resulting equation is

$$F\left(\frac{az' - cx'}{z' - c}, \frac{bz' - cy'}{z' - c}\right) = 0 \ldots\ldots\ldots (8).$$

If the director be the circle of which the equations are

$$z = 0, \quad x^2 + y^2 = r^2,$$

then the equation to an oblique cone with a circular base is

$$(az' - cx')^2 + (bz' - cy')^2 = r^2 (z' - c)^2 \ldots\ldots (9).$$

(205) Let the cone circumscribe the ellipsoid

$$Ax^2 + A'y^2 + A''z^2 = 1 \ldots\ldots\ldots\ldots (10);$$

then by equation (17) of Art. (165) the curve of contact is given by the combination of (10) with

$$Aax + A'by + A''cz = 1 \text{ .... } (11),$$

a, b, c being the co-ordinates of the vertex of the cone.

The equations to the generator may be written

$$\frac{x'-x}{a-x} = \frac{y'-y}{b-y} = \frac{z'-z}{c-z} \text{ .... } (12).$$

On subtracting (10) from (11), and multiplying each term by the corresponding member of (12), we have

$$Axx' + A'yy' + A''zz' = 1 \text{ .... } (13),$$

and the system (11) and (13) may be used instead of (10) and (11) for the elimination of x, y, z. Multiply the numerator and denominator of each member of (12) by $Aa, A'b, A''c$ respectively, and add, then by Theorem I. of Art. (22) each member of (12) is equal to

$$\frac{Aa(x'-x) + A'b(y'-y) + A''c(z'-z)}{Aa(a-x) + A'b(b-y) + A''c(c-z)} = \frac{Aax' + A'by' + A''cz' - 1}{Aa^2 + A'b^2 + A''c^2 - 1},$$

in virtue of (10) and (12). Again, doing the same with $Ax', A'y', A''z'$, we have each ratio equal to

$$\frac{Ax'(x'-x) + A'y'(y'-y) + A''z'(z'-z)}{Ax'(a-x) + A'y'(b-y) + A''z'(c-z)} = \frac{Ax'^2 + A'y'^2 + A''z'^2 - 1}{Aax' + A'by' + A''cz' - 1}$$

and hence equating these ratios, we have

$$(Aa^2 + A'b^2 + A''c^2 - 1)(Ax'^2 + A'y'^2 + A''z'^2 - 1) = (Aax' + A'by' + A''cz' - 1)^2 \text{ .... } (14),$$

as the equation to the cone.

Let the distance of (a, b, c) from the origin be r, and its direction-cosines therefore $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$, and let ρ be the portion of r intercepted between the surface and the origin, or the diameter to the plane of contact, then at the extremity of ρ

$$x = a\frac{\rho}{r}, \quad y = b\frac{\rho}{r}, \quad z = c\frac{\rho}{r},$$

which being substituted in (10) give

$$Aa^2 + A'b^2 + A''c^2 = \frac{r^2}{\rho^2},$$

so that the equation to the circumscribing cone may be written

$$(r^2 - \rho^2)(Ax'^2 + A'y'^2 + A''z'^2 - 1) = \rho^2(Aax' + A'by' + A''cz' - 1)^2.$$

Conoidal Surfaces.

(206) Conoidal surfaces are generated by the motion of a straight line which passes through a fixed axis and remains always perpendicular to it.

Let the equations to the axis be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \;\ldots\ldots\ldots\ldots (1);$$

then, since the generator is perpendicular to this axis, it lies in the plane

$$lx+my+nz=\delta \;\ldots\ldots\ldots\ldots\ldots (2),$$

and also in some plane passing through the axis, the equation to which may be written

$$n(y-b)-m(z-c)=\delta_1\{n(x-a)-l(z-c)\}\ldots(3),$$

and δ and δ_1 being the arbitrary parameters, there must be some relation between them, as $\delta=\phi(\delta_1)$;

whence we have

$$lx+my+nz=\phi\left\{\frac{n(y-b)-m(z-c)}{n(x-a)-l(z-c)}\right\},$$

as the general equation to conoidal surfaces. If we take the axis of z as the axis of the surface, we have $l=0$, $m=0$, $n=1$, $a=0$, $b=0$, $c=0$, so that in this case the equation becomes

$$z=\phi\left(\frac{y}{x}\right);$$

showing that the ordinate parallel to the axis of the conoid is equal to a homogeneous function of 0 dimensions in x and y. This is the class of surfaces treated of in Art. (186).

(207) Let the director be the plane curve determined by the equations

$$lx+my+nz=\delta, \quad F(x, y, z)=0,$$

and take the axis of z as the axis of the surface; then the equations to the generator are

$$\frac{x'}{x}=\frac{y'}{y}, \quad z'=z,$$

x', y', z' being the current co-ordinates of the generator: from these we have

$$\frac{x'}{x}=\frac{y'}{y}=\frac{lx'+my'}{lx+my}=\frac{lx'+my'}{\delta-nz}=\frac{lx'+my'}{\delta-nz'},$$

whence $\quad x = x' \dfrac{\delta - nz'}{lx' + my'}, \quad y = y' \dfrac{\delta - nz'}{lx' + my'}, \quad z = z',$

which values, substituted in $F(x, y, z) = 0$, give

$$F\left(x' \frac{\delta - nz'}{lx' + my'}, \quad y' \frac{\delta - nz'}{lx' + my'}, \quad z'\right) = 0$$

as the equation to the conoid.

(208) Let the director be the circle of which the equations are

$$x = b, \quad y^2 + z^2 = a^2;$$

then the equation to the conoid is

$$\frac{b^2 y'^2}{x'^2} + z'^2 = a^2, \quad \text{or} \quad x'^2 z'^2 = a^2 x'^2 - b^2 y'^2.$$

This surface is called the cono-cuneus of Wallis, that mathematician having been the first who conceived it and investigated its properties. It is easy to see that planes $z' = \pm c$ parallel to (x', y') cut it in two straight lines so long as c is less than a, but when c is greater than this value the planes do not meet it, so that the surface is limited within the space included between the planes $z = +a$, $z = -a$.

(209) To find the equation to the tangent plane and the nature of the contact in the cono-cuneus. Dropping the accents we may write the equation to the surface as

$$(a^2 - z^2)\, x^2 - b^2 y^2 = 0;$$

then, x', y', z' being the current co-ordinates of the tangent plane, its equation is $\quad x^2 zz' - (a^2 - z^2)\, xx' + b^2 yy' = x^2 z^2.$

To find the lines in which this meets the surface, we must combine these equations with

$$(a^2 - z'^2)\, x'^2 - b^2 y'^2 = 0.$$

The second equation gives

$$b^2 yy' = x^2 (z^2 - zz') + (a^2 - z^2)\, xx';$$

from which, on eliminating y and y' by means of the first and third equations, we have

$$(a^2 - z^2)^{\frac{1}{2}} (a^2 - z'^2)^{\frac{1}{2}}\, x' = xz\,(z - z') + (a^2 - z^2)\, x';$$

whence, by squaring both sides and omitting the terms which destroy each other, we find

$$(a^2 - z^2)(z^2 - z'^2)\, x'^2 = x^2 z^2 (z - z')^2 + 2xz\,(a^2 - z^2)(z - z')\, x'.$$

This may be put in the form

$$(z - z') \left[(a^2 - z^2)\{x'^2(z + z') - 2xzx'\} - x^2z^2(z - z')\right] = 0,$$

which it is easy to see may be split into the two factors

$$z - z' = 0,$$

$$(a^2 - z^2)\{x'^2(z + z') - 2xzx'\} - x^2z^2(z - z') = 0.$$

The former, combined with the equation to the tangent plane, gives a straight line which is in fact the generator, the other gives a curve of the third order. Hence the tangent plane cuts the surface in two lines, one of the first, the other of the third order.

(210) Let the director of the conoid be the helix of which the equations are

$$x = a \cos nz, \quad y = a \sin nz.$$

The equations to the generator are

$$\frac{x'}{x} = \frac{y'}{y}, \quad z' = z,$$

whence $x'y - y'x = 0$, or $x' \sin nz - y' \cos nz = 0$;

and the equation to the conoid is, therefore,

$$x' \sin nz' - y' \cos nz' = 0.$$

This remarkable surface, which may be called the "skew screw surface," to distinguish it from another of which we shall speak presently, is that which forms the under surface of a spiral staircase, and is consequently one which is frequently presented to the eye, and is also easily constructed.

To find the equation to the tangent plane; the equation to the surface being $x \sin nz - y \cos nz = 0$,

that to the tangent plane is

$$\sin nz\,(x' - x) - \cos nz\,(y' - y) + n\,(x \cos nz + y \sin nz)(z' - z) = 0;$$

but since $\dfrac{x}{\cos nz} = \dfrac{y}{\sin nz}$ this may be reduced to

$$yx' - xy' + n\,(x^2 + y^2)(z' - z) = 0.$$

(211) Let the conoid circumscribe the sphere

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2 \;\ldots\ldots\ldots\; (1),$$

the equations to the generator being

$$u = x'y - y'x = 0, \quad v = z' - z = 0,$$

the equation of Art. (193) becomes

$$x\frac{dF}{dx}+y\frac{dF}{dy}=0,$$

so that in this case the director consists of (1) combined with

$$x(x-a)+y(y-\beta)=0\ldots\ldots\ldots\ldots(2).$$

The equations to the generator

$$\frac{x'}{x}=\frac{y'}{y},\quad z'=z$$

give $\frac{x'}{x}=\frac{y'}{y}=\frac{x'(x-a)+y'(y-\beta)}{x(x-a)+y(y-\beta)}=\frac{x'(x-a)+y'(y-\beta)}{0}$

in virtue of (2), consequently

$$x'(x-a)+y'(y-\beta)=0;$$

or $$\frac{x-a}{y'}=\frac{y-\beta}{-x'}=\frac{(x-a)y'-(y-\beta)x'}{x'^2+y'^2}=\frac{\beta x'-ay'}{x'^2+y'^2};$$

whence $$x-a=y'\frac{\beta x'-ay'}{x'^2+y'^2},\quad y-\beta=-x'\frac{\beta x'-ay'}{x'^2+y'^2};$$

and, on substituting in (1), we have

$$(\beta x'-ay')^2=(x'^2+y'^2)\{r^2-(z'-\gamma)^2\}$$

as the required equation.

Skew Surfaces having more than one Director.

(212) In the families of surfaces which we have been considering there is only one director, so that the requisite eliminations are sufficiently simple; we shall now give some examples of the investigation of the equations of ruled surfaces which have more than one director. The equations to the generator, when put in the form

$$\frac{x-a}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

appear to contain six parameters, but as they can always be reduced to the form

$$x=az+p,\quad y=bz+q,$$

we see that there are really only four independent parameters. The condition that the generator shall pass through a director gives one relation between the constants, and if there be three directors we have thus three equations involving the constants

which, combined with the two equations to the generator, give us five equations, between which the four parameters may be eliminated, and an equation obtained between x, y, z, which is the equation to the surface.

(213) Find the equation to the surface generated by the motion of a straight line which passes through the circumference of a circle, and also through two straight lines at right angles to each other and parallel to the plane of the circle, their shortest distance passing through its centre. Taking the centre of the circle as origin and the axis of x coinciding with the shortest distance, the equations to the directors are

(1) $x = 0,\ y^2 + z^2 = a^2$, (2) $z = 0,\ x = b$, (3) $y = 0,\ x = -b$.

The equations to the generating line are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n};$$

As it intersects (2), we have

$$\frac{x-b}{l} = \frac{z}{n} \quad \text{and} \quad \frac{n}{l} = \frac{z}{x-b}.$$

As it intersects (3)

$$\frac{x+b}{l} = \frac{y}{m} \quad \text{and} \quad \frac{m}{l} = \frac{y}{x+b}.$$

As it meets (1), α, β, γ may be supposed to satisfy these equations, and as $\alpha = 0$, the equations to the generator give

$$\gamma = z - \frac{n}{l}x = z - \frac{xz}{x-b} = -\frac{bz}{x-b},$$

$$\beta = y - \frac{m}{l}x = y - \frac{xy}{x+b} = \frac{by}{x+b},$$

so that
$$\frac{y^2}{(x+b)^2} + \frac{z^2}{(x-b)^2} = \frac{a^2}{b^2}$$

is the required equation.

(214) Find the equation to a surface defined in the following manner. On the opposite sides AB, CD (fig. 28) of an oblique parallelogram are described two semicircles having their planes perpendicular to that of the parallelogram; the surface is traced out by a straight line which rests on these semicircles and on a straight line MON passing through the centre of the

parallelogram and perpendicular to the plane of the circles. Take the plane of the parallelogram as the plane of (x, y), and the straight line MON as the axis of y: then if O be the origin, the equations to the three directors may be written as

$$x = 0, \quad z = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$y = -b, \quad (x-a)^2 + z^2 = r^2 \ldots\ldots\ldots\ldots(2),$$

$$y = b, \quad (x+a)^2 + z^2 = r^2 \ldots\ldots\ldots\ldots(3).$$

We may at once express the condition that the generator passes through the director (1) by writing its equations

$$x = \alpha(y - \beta), \quad z = \gamma(y - \beta) \ldots\ldots\ldots(4).$$

The conditions that it shall rest on (2) and (3) give the equations

$$\{\alpha(b+\beta) + a\}^2 + \gamma^2(b+\beta)^2 = r^2 \ldots\ldots\ldots(5),$$

$$\{\alpha(b-\beta) + a\}^2 + \gamma^2(b-\beta)^2 = r^2 \ldots\ldots\ldots(6);$$

from which, by subtraction, we have

$$\beta(b\alpha^2 + a\alpha + b\gamma^2) = 0 \ldots\ldots\ldots\ldots(7).$$

This equation may be satisfied by $\beta = 0$, but that would correspond to the generator passing always through the origin, in which case the surface would be a cone; we take, therefore, the other solution $$b\alpha^2 + a\alpha + b\gamma^2 = 0 \ldots\ldots\ldots\ldots\ldots(8);$$

which, being substituted in (5), reduces it to

$$(b^2 - \beta^2)\,a\alpha = b(r^2 - a^2) \ldots\ldots\ldots\ldots(9).$$

Between the four equations (4), (8), (9) we may eliminate the three parameters α, β, γ, and we obtain as the final equation, which is that of the surface,

$$\{axy + b(x^2 + z^2)\}^2 = b^2r^2x^2 + b^2(r^2 - a^2)z^2.$$

(215) When there are only two directors, some other geometrical relation must be assigned in order that the motion of the generator may be completely regulated. The condition most usually taken is that the generator shall remain parallel to a fixed plane; in this case we may find a general equation for the class of surfaces so generated. Taking the plane of (x, y) as parallel to the fixed plane, the equations to the generator are

$$z = \alpha, \quad y = \beta x + \gamma.$$

Now the conditions of the generator resting on two directors give two relations between the parameters

$$\psi(\alpha, \beta, \gamma) = 0, \quad \psi_1(\alpha, \beta, \gamma) = 0;$$

from which we may find values for two of them in terms of the third, as $\beta = \phi(\alpha), \quad \gamma = \phi_1(\alpha)$.

Eliminating then α, β, γ between these equations and those to the generator, we have

$$y = x\phi(z) + \phi_1(z)$$

as the general equation to ruled surfaces, generated by the motion of a straight line which is always parallel to the plane of (x, y).

(216) As an example of such surfaces take the following. A sphere touches a circle of equal radius in the centre; the surface is traced out by a straight line which, always parallel to a plane perpendicular to that of the circle, touches the sphere and passes through the circumference of the circle. Take the plane of the circle as that of (x, y), its centre being the origin, and the plane of (x, z) parallel to the fixed plane; the equations to the generator are $x = \alpha, \quad y = \beta z + \gamma \ldots\ldots\ldots\ldots\ldots (1)$;

those to the directors are

$$x^2 + y^2 = a^2, \quad z = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

$$x^2 + y^2 + z^2 - 2az = 0, \quad \beta y + z - a = 0 \ldots\ldots (3),$$

the last equation being the condition that the generator shall touch the sphere. The condition of the generator passing through (2) gives the equation

$$\alpha^2 + \gamma^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

the condition that it shall pass through (3) gives

$$\beta(a^2\beta + 2a\gamma - \beta\gamma^2) = 0.$$

This may be satisfied by $\beta = 0$, which would lead to the equation of a cylinder perpendicular to the plane of (x, y), a solution which evidently satisfies the geometrical conditions. The other factor gives $a^2\beta + 2a\gamma - \beta\gamma^2 = 0 \ldots\ldots\ldots\ldots\ldots (5)$.

Between (1), (4) and (5) we may eliminate α, β, γ, and we obtain as the equation to the surface

$$x^4y^2 = (a^2 - x^2)(x^2 - 2az)^2.$$

Developable Surfaces.

(217) In Art. (197) we showed that any developable surface may be considered as generated by the motion of a straight line which is always a tangent to the cuspidal edge. Let the equations to the cuspidal edge be

$$x = \phi(z), \quad y = \psi(z);$$

then, if x', y', z' be the current co-ordinates of the tangent, its equations may be written

$$x' - \phi(z) = \phi'(z)(z' - z),$$
$$y' - \psi(z) = \psi'(z)(z' - z);$$

between these two equations we may eliminate z, and we then obtain an equation between x', y', z', which is the equation to the surface.

(218) Thus if the cuspidal edge be the helix

$$x = a\cos nz, \quad y = a\sin nz,$$

the equations to the tangent are

$$x' - a\cos nz = -na\sin nz\,(z' - z),$$
$$y' - a\sin nz = na\cos nz\,(z' - z).$$

To eliminate z, multiply by $\cos nz$, $\sin nz$ and add, then

$$x'\cos nz + y'\sin nz = a;$$

squaring both equations and adding, we have

$$x'^2 + y'^2 - a^2 = n^2a^2(z' - z)^2;$$

whence $$z = z' \mp \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{na},$$

and therefore

$$x'\cos\left\{nz' \mp \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a}\right\} + y'\sin\left\{nz' \mp \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a}\right\} = a,$$

which is the equation to the surface.

(219) This surface is called the "developable screw surface," to distinguish it from that of which we treated in Art. (210). It is obvious from the form of the equation that the surface falls back from the cuspidal edge; for as the equation becomes impossible when $$x'^2 + y'^2 < a^2,$$

no part of the surface lies within the cylinder on which the helix is traced. If in the equation to the surface we make $z' = 0$, we have

$$x' \cos \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a} \mp y' \sin \frac{(x'^2 + y'^2 - a^2)^{\frac{1}{2}}}{a} = a,$$

showing that the section of the surface by the plane of (x', y') is the involute of the circle.

If we take the upper of the double signs in the equation to the surface, we find that the direction-cosines of the tangent plane are proportional to

$$\cos\left\{nz - \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} - \frac{x}{a}, \quad \sin\left\{nz - \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} - \frac{y}{a}$$

and $$n (x^2 + y^2 - a^2)^{\frac{1}{2}};$$

from which we see that the cosine of the inclination of the tangent plane to the plane of (x, y) is

$$\frac{na}{(1 + n^2a^2)^{\frac{1}{2}}},$$

and is therefore constant.

(220) Let the cuspidal edge be the equable spherical spiral of which the equations are

$$y^2 = 2rx - x^2, \quad z^2 = 2rx.$$

The equations to the tangent are

$$yy' - (r - x) x' = rx, \quad zz' = r (x + x');$$

if we take the values of y and z derived from the second pair of equations and substitute them in the first pair, we obtain the two following quadratic equations in x,

$$\{(r - x')^2 + y'^2\} x^2 + 2r (rx' - x'^2 - y'^2) x + r^2x'^2 = 0,$$

$$r^2x^2 + 2r (rx' - z'^2) x + r^2x'^2 = 0.$$

Between these two equations we may eliminate x, and the result is, after simplifying and dividing by $r^4x'^2$,

$$x'^2 (x'^2 + y'^2 - 2rx')^2$$
$$= 4 (z'^2 - x'^2 - y'^2) \{(rx' - z'^2)(r^2 - rx') + (r^2 + rx' - z'^2)(x'^2 + y'^2 - rx')\},$$

which is the required equation to the developable surface of which the edge is the equable spherical spiral.

Surfaces of Revolution.

(221) Of surfaces admitting of a circle as generator we shall consider those of revolution alone. These may be defined as generated by the motion of a circle of variable radius, the centre of which moves along a straight line to which the plane of the circle is always perpendicular: the circumference of the circle passes always through the curve which, by its revolution round the axis, generates the surface, and which in fact is the director.

Let the equations to the fixed straight line, or axis, as it is called, be
$$\frac{x-a}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$
then the generator in any position may be determined by the intersection of a sphere of which the centre is at the point a, β, γ with a plane perpendicular to the axis. The equation to such a plane is in the form
$$lx + my + nz = c,$$
and that to the sphere is
$$(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2 = c_1;$$
in this case the parameters are c and c_1, and if the relation between them be
$$c_1 = \phi(c),$$
the general equation to surfaces of revolution is
$$(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2 = \phi(lx + my + nz).$$
If we take the axis of z as the axis of revolution, $l = 0$, $m = 0$, $n = 1$, and if (a, β, γ) be taken as origin, the equation becomes
$$x^2 + y^2 + z^2 = \phi(z),$$
which is equivalent to
$$x^2 + y^2 = \phi(z), \quad \text{or } z = \psi(x^2 + y^2).$$
This form may be derived immediately by considering the generator as determined by the intersection of a right circular cylinder with a plane.

(222) To find the surface generated by the revolution of a straight line round an axis which it does not meet. The director in this case is a straight line: let its equations be
$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n};$$

then, if the axis of z be taken as the axis of revolution, the equations to the generating circle are

$$x'^2 + y'^2 = r^2, \quad z' = \gamma,$$

and, by Art. (192), we may write them

$$x'^2 + y'^2 = x^2 + y^2, \quad z' = z.$$

Now $$x = a + \frac{l}{n}(z - c), \quad y = b + \frac{l}{m}(z - c),$$

and therefore, eliminating x, y and z, we have

$$x'^2 + y'^2 - \frac{l^2 + m^2}{n^2}(z' - c)^2 - 2\frac{la + mb}{n}(z' - c) = a^2 + b^2$$

as the equation to the surface of revolution, which is evidently a hyperboloid of revolution, round the axis of z.

(223) Let the surface be generated by the revolution of a circle round an axis lying in the plane of the circle. Taking, as before, the axis of z as that of revolution, and supposing the director to be in the plane of (x, z), its equations may be written as $$(x - b)^2 + z^2 = a^2, \quad y = 0,$$
a being the radius of the circle and b the distance of its centre from the origin; and those to the generating circle are

$$x^2 + y^2 = r^2, \quad z = \gamma,$$

$$\text{or} \quad x'^2 + y'^2 = x^2 + y^2, \quad z' = z.$$

Hence, as $$y = 0, \quad x = (x'^2 + y'^2)^{\frac{1}{2}},$$
and therefore $$\{(x'^2 + y'^2)^{\frac{1}{2}} - b\}^2 + z'^2 = a^2 \ldots\ldots\ldots\ldots (1).$$
On clearing this equation of radicals, and, for convenience, dropping the accents, this equation becomes

$$(x^2 + y^2 + z^2 + b^2 - a^2)^2 - 4b^2(x^2 + y^2) = 0 \ldots\ldots (2),$$

which is of the fourth degree. It is easy to see from the form (1) that the surface is limited between the planes $z = +a$ and $z = -a$, and that its form differs according as b is greater or less than a. If $b > a$, no part of the surface approaches nearer to the surface than a distance equal to $b - a$, so that the axis never meets the surface. In this case it is the bounding surface of an anchor ring. If $b < a$ there is an interior sheet which

meets the external sheet at two points in the axis, the distances of which from the origin are

$$\pm (a^2 - b^2)^{\frac{1}{2}}.$$

All sections of this surface made by planes perpendicular to the axis are pairs of concentric circles. For if $z = h$ be the equation to such a plane, we must have $h < a$, and hence the equation to the projection of the section on the plane of (x, y) is, if we put $a^2 - h^2 = c^2$,

$$(x^2 + y^2 + b^2 - c^2)^2 - 4b^2(x^2 + y^2) = 0,$$

or

$$(x^2 + y^2)^2 - 2(b^2 + c^2)(x^2 + y^2) + (b^2 - c^2)^2 = 0,$$

which is decomposable into the two equations

$$x^2 + y^2 - (b + c)^2 = 0, \quad x^2 + y^2 - (b - c)^2.$$

All sections made by planes parallel to the axis are lemniscates of various kinds: those made by planes at a distance less than $b - a$ from the axis will consist of two distinct ovals, those made by planes at a greater distance will be one continuous curve. If the cutting plane be $y = b - a$, the equation to the section becomes

$$(x^2 + z^2)^2 = 4b\,\{ax^2 - (b - a)z^2\},$$

which is the lemniscate of Bernoulli.

(224) If the surface of revolution, of which the axis is the axis of z, is to circumscribe a given surface $F(x, y, z) = 0$, then, by Art. (193), the equations to the director are

$$F(x, y, z) = 0, \quad x\frac{dF}{dy} - y\frac{dF}{dx} = 0.$$

Let the given surface be the ellipsoid of revolution

$$\frac{(x - \alpha)^2 + (y - \beta)^2}{a^2} + \frac{(z - \gamma)^2}{c^2} = 1,$$

then this equation is to be combined with

$$x(y - \beta) - y(x - \alpha) = 0, \quad \text{or} \quad \frac{x}{\alpha} = \frac{y}{\beta}.$$

The equations to the generator may be assumed to be

$$x'^2 + y'^2 = x^2 + y^2, \quad z' = z;$$

hence

$$\frac{x}{\alpha} = \frac{y}{\beta} = \pm\frac{(x^2 + y^2)^{\frac{1}{2}}}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} = \pm\frac{(x'^2 + y'^2)^{\frac{1}{2}}}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}$$

and
$$x - a = \frac{a}{(a^2+\beta^2)^{\frac{1}{2}}}\{\pm(x'^2+y'^2)^{\frac{1}{2}} - (a^2+\beta^2)^{\frac{1}{2}}\}$$

$$y - \beta = \frac{\beta}{(a^2+\beta^2)^{\frac{1}{2}}}\{\pm(x'^2+y'^2)^{\frac{1}{2}} - (a^2+\beta^2)^{\frac{1}{2}}\}.$$

Substituting these values and that of z in the equation to the ellipsoid of revolution, we have

$$\frac{\{(a^2+\beta^2)^{\frac{1}{2}} \mp (x'^2+y'^2)^{\frac{1}{2}}\}^2}{a^2} + \frac{(z'-\gamma)^2}{c^2} = 1,$$

as the equation to the surface of revolution, which, having the axis of z as its axis, circumscribes the given ellipsoid.

CHAPTER XI.

ENVELOPS TO SURFACES.

(225) If the equation to a surface involve an arbitrary constant or parameter, we may suppose this quantity to receive a succession of values, in consequence of which the surface will change in form, or in position, or in both; then the surface which touches the moveable surface in all its positions and forms is called the *envelop* to the series of surfaces which are formed by assigning all possible values to the parameter.

Let the equation to the surface involving the arbitrary constant or parameter a, be

$$f(x, y, z, a) = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

Then if a become $a + h$, the equation

$$f(x, y, z, a + h) = 0 \ldots\ldots\ldots\ldots\ldots (2)$$

represents the same surface in another form or position, which may be brought as near to the original one as we please by making h as small as we please. Now if between (1) and (2) we eliminate a, we obtain the equation to a surface

$$F(x, y, z, h) = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

which passes through the intersection of (1) and (2) independently of any value of a. But by diminishing h we may make (1) and (2) approach as nearly as we please, and the limit to which (3) tends, as h is diminished, is

$$F(x, y, z) = 0 \ldots\ldots\ldots\ldots\ldots (4).$$

But we should have arrived at the same result if we considered the limit of the intersection of

$$f(x, y, z, a) = 0 \quad \text{and} \quad f(x, y, z, a - h) = 0\,;$$

and therefore the equation (4) is the limit of the equation to the surface which passes through the intersection of $f(x, y, z, a) = 0$

with $f(x, y, z, a + h) = 0$ and $f(x, y, z, a - h) = 0$, as h is indefinitely diminished, or the equation

$$F(x, y, z) = 0$$

represents a surface which is the limit of the surfaces passing through two lines which ultimately coincide, and it is therefore the equation to a surface touching the surface $f(x, y, z, a) = 0$, in which these lines lie; and as it is independent of a, this is true for all positions of the moveable surface; and consequently it is the equation to the envelop.

(226) Now if we put $u = f(x, y, z, a) = 0$, we have

$$f(x, y, z, a + h) = u + \frac{du}{da} h + \frac{d^2u}{da^2} \frac{h^2}{1.2} + \&c. = 0,$$

which, as $u = 0$, is equivalent to

$$h \left\{\frac{du}{da} + \frac{h}{1.2} \frac{d^2u}{da^2} + \&c.\right\} = 0.$$

This gives either $h = 0$, or

$$\frac{du}{da} + \frac{h}{1.2} \frac{d^2u}{da^2} + \&c. = 0.$$

The former would imply that a does not receive any increment, or that it is absolutely constant, which is inconsistent with our assumption. We must therefore take the other equation

$$\frac{du}{da} + \frac{h}{1.2} \frac{d^2u}{da^2} + \&c. = 0,$$

which, when h is indefinitely diminished, has for its limit

$$\frac{du}{da} = 0.$$

Hence, instead of eliminating a between (1) and (2), and then making $h = 0$ in the result, we shall obtain the same equation by eliminating a between

$$u = 0, \quad \text{and} \quad \frac{du}{da} = 0,$$

and the result is the equation to the envelop.

Since, by what has been said, the envelop passes through the intersection of two consecutive surfaces, when h is indefinitely diminished, it is frequently called the *locus of the ultimate intersections* of a series of surfaces described after a given law.

(227) Before proceeding further we may illustrate the preceding theory by a simple example. Let it be required to find the envelop of a series of spheres determined by the equation

$$(x-a)^2 + y^2 + z^2 - n^2a^2 = 0 \ldots\ldots\ldots\ldots (1),$$

in which a is the variable parameter. These spheres will change both in form and in position with the variation of a, the radius being in fact proportional to the distance of the centre from a fixed point. Differentiating (1) with respect to a, and equating to zero, we have

$$x - a + n^2a = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From this we have $a = \dfrac{x}{1-n^2}$, and by means of this eliminating a from (1), we have

$$\frac{n^2x^2}{n^2-1} + y^2 + z^2 = 0$$

as the equation to the envelop, which is evidently a right cone, if $n < 1$ and is reduced to a point if $n > 1$.

(228) The two equations

$$u = 0, \quad \frac{du}{da} = 0$$

may be viewed in a somewhat different light: for if we consider a as constant in both, they each represent a surface, and taken together they represent a line, which is in fact the line in which the envelop touches the surface $u = 0$, corresponding to the particular value of a. Now if this line be supposed to change in position or form in consequence of the variation of a, it will, according to the theory developed in the previous chapter, trace out a surface which is of course the same as the envelop. This curve has been named by Monge the *characteristic*, because it determines the family of the envelop exactly as in the generation of a surface by a line, which is the intersection of two surfaces, the family of the surface generated is determined by the line of intersection, independently of the particular law of variation of the parameter.

(229) In the example which we have just adduced, if the equation to the sphere be called $u = 0$, we have

$$\frac{du}{da} = x - (1 - n^2)\,a = 0:$$

this is the equation to a plane perpendicular to the axis of x, and the *characteristic* is the circle determined by the intersection of this plane with the sphere $u = 0$. From the equation to the plane it is obvious that whatever a may be, the plane does not pass through the centre of the sphere, and consequently the characteristic is in this case a *small* circle of the sphere.

(230) It is easy to shew that the surface $u = 0$, in which a is supposed to be constant, is always touched by the surface $u = 0$, in which a is supposed to be a function of x, y, z, determined by the equation

$$\frac{du}{da} = 0.$$

For the position of the tangent plane of the first surface is determined by the quantities

$$\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz},$$

which are proportional to the direction-cosines. In the second surface, where a is a function of x, y, z in virtue of the equation $\frac{du}{da} = 0$, the tangent plane is determined by the quantities

$$\frac{du}{dx} + \frac{du}{da}\frac{da}{dx}, \quad \frac{du}{dy} + \frac{du}{da}\frac{da}{dy}, \quad \frac{du}{dz} + \frac{du}{da}\frac{da}{dz}:$$

and in consequence of the condition $\frac{du}{da} = 0$, these quantities are reduced to those which precede; therefore the tangent planes coincide, and the surfaces touch each other.

(231) Again, the three surfaces of which the equations are

$$f(x, y, z, a) = 0, \quad f(x, y, z, a + h) = 0, \quad f(x, y, z, a - h) = 0$$

will in general intersect each other in a point, the co-ordinates of which may be found from these equations. If we write them in the form

$$u = 0, \quad u + h\frac{du}{da} + \frac{h^2}{1.2}\frac{d^2u}{da^2} + \&c. = 0,$$

$$u - h\frac{du}{da} + \frac{h^2}{1.2}\frac{d^2u}{da^2} - \&c. = 0,$$

they are equivalent to the three

$$u = 0, \quad h\left\{\frac{du}{da} + \frac{h}{1.2}\frac{d^2u}{da^2} + \&c.\right\} = 0,$$

$$h^2\left\{\frac{d^2u}{da^2} + \frac{h^2}{3.4}\frac{d^4u}{da^4} + \&c.\right\} = 0.$$

On rejecting the factors h and h^2 and taking the limit as h is indefinitely diminished, we find the equations

$$u = 0, \quad \frac{du}{da} = 0, \quad \frac{d^2u}{da^2} = 0,$$

as those for determining the point which is common to three consecutive positions of the surface, or, what is equivalent, to two consecutive positions of the characteristic. A series of these points must in general exist corresponding to successive values of a, and the equation to the line in which they lie may be found by eliminating a between any two pair of these three equations, so as to obtain two equations in x, y, z independent of a, representing two surfaces the intersection of which determine the line in question.

It is clear that this line must lie on the envelop, because we may first eliminate a between $u = 0$ and $\frac{du}{da} = 0$, which gives us the envelop, and then eliminate a again between either of these equations and $\frac{d^2u}{da^2} = 0$, so as to obtain the second equation. This line, which is a very remarkable one on the enveloping surface, is called the *edge* of the envelop.

(232) It is to be observed that such a line does not always exist, for the second elimination of a may lead to an equation which cannot be satisfied by any possible values of x, y, z, shewing that the successive characteristics do not meet. It may also happen that the combination of the three equations $u = 0$, $\frac{du}{da} = 0$, $\frac{d^2u}{da^2}$ leads to one or more separate values of x, y, and z, in which case the edge is reduced to one or more points. Thus in cones the edge is reduced to a point which is the vertex of the cone. As in the case of the enveloping and enveloped surfaces, it is

easy to shew that we may obtain the same values for $\frac{dz}{dx}$ and $\frac{dz}{dy}$, whether they be derived from the equations $u = 0$ and $\frac{du}{da} = 0$ on the supposition that a is constant, or that it is variable subject to the condition $\frac{d^2u}{da^2} = 0$: hence any characteristic and the edge of the envelop touch each other at the point where they meet.

(233) If the equation to the moveable surface contain n parameters connected by $n - 1$ equations of condition, it is convenient, instead of expressing $n - 1$ of the parameters in terms of the n^{th} by means of the equations of condition, and so reducing the question to a case involving one parameter, to consider them all as variable, subject to the equations of condition, and to eliminate by the method of indeterminate multipliers, as will be best seen in examples.

(234) If the equation to the surface contain several independent parameters a, b, c, &c., so as to be in the form

$$u = f(x, y, z, a, b, c \ldots) = 0,$$

it is easy to see, by following the same method as in the case of one parameter, that the surface found by eliminating a, b, c, &c. between the equations

$$u = 0, \quad \frac{du}{da} = 0, \quad \frac{du}{db} = 0, \quad \frac{du}{dc} = 0, \text{ \&c.}$$

touches the moveable surface in all its forms and positions, and is therefore its envelop. It cannot however be considered as the locus of ultimate intersections, since, in consequence of there being several independent parameters, there are an infinite number of positions of the moveable surface consecutive to any given one: for the same reason the theory of characteristics has here no place. Thus, for example, let the moveable surface be the sphere

$$u = (x - \alpha)^2 + (y - \beta)^2 + z^2 - r^2 = 0,$$

in which α, β are the variable parameters. We have

$$\frac{du}{d\alpha} = -2(x - \alpha) = 0, \quad \frac{du}{d\beta} = -2(y - \beta) = 0,$$

by means of which, eliminating α and β, we find

$$z^2 - r^2 = 0$$

as the equation to the envelop. This evidently indicates two planes parallel to the plane of (x, y) at equal distances: it is obvious that they can touch each position of the moveable sphere each in one point only.

We now proceed to more extended applications of the preceding theory.

(235) *Developable surfaces.* If the equation to a plane involve only one arbitrary constant, the others being assigned functions of it, we may, for the sake of symmetry, assume that the three coefficients are all functions of one variable. If then the equation to the plane be

$$ax + by + cz = 1 \ldots\ldots\ldots\ldots\ldots (1),$$

where a, b, c are functions of a single variable α, the plane will, in consequence of the variation of α, assume a series of different positions, and we may find the equation to the surface which is the envelop of these planes. This will be found by eliminating α between the equation to the plane and

$$\frac{da}{d\alpha} x + \frac{db}{d\alpha} y + \frac{dc}{d\alpha} z = 0 \ldots\ldots\ldots\ldots (2).$$

The combination of (1) and (2) gives the equation to the characteristic which is in this case a straight line, as it is determined by the intersection of two planes (1) and (2). It is easy to see that the envelop in this case is a developable surface, for any two consecutive characteristics being in the same plane must intersect, and consequently the surface may be considered as generated by the motion of a straight line, the consecutive positions of which intersect, and therefore, by Art. (160), it is a developable surface.

The equations to the edge of the envelop or developable surface are found by eliminating α between (1), (2), and

$$\frac{d^2a}{d\alpha^2} x + \frac{d^2b}{d\alpha^2} y + \frac{d^2c}{d\alpha^2} z = 0 \ldots\ldots\ldots\ldots (3).$$

(236) We see therefore that a developable surface is the locus of the ultimate intersections of a plane which involves only one variable parameter. Any plane therefore which moves subject

to this condition will produce a developable surface. It is obvious that this condition is fulfilled in the case of a normal or an osculating plane to a curve of double curvature, since the equations

$$(x'-x)\,dx+(y'-y)\,dy+(z'-z)\,dz=0,$$

$$(x'-x)(dy\,d^2z-dz\,d^2y)+(y'-y)(dz\,d^2x-dx\,d^2z)$$
$$+(z'-z)(dx\,d^2y-dy\,d^2x)=0,$$

can be considered as containing only one independent quantity of the three x, y, z, since these are connected by the two relations which are the equations to the curve.

(237) If a developable surface be generated by the ultimate intersections of the successive osculating planes to a curve of double curvature, the *characteristic* is the tangent to the curve, so that the surface may be conceived as generated by the motion of a straight line, which constantly touches the curve. For the equation to the osculating plane may be put under the form

$$\{(y'-y)dz-(z'-z)dy\}d^2x+\{(z'-z)dx-(x'-x)dz\}d^2y$$
$$+\{(x'-x)dy-(y'-y)dx\}d^2z=0.$$

Differentiating this, considering x, y, z as the variables, we have

$$\{(y'-y)dz-(z'-z)dy\}d^3x-\{(z'-z)dx-(x'-x)dz\}d^3y$$
$$+\{(x'-x)dy-(y'-y)dx\}d^3z=0.$$

The characteristic is determined by the combination of these two equations, which involves the relations

$$\frac{x'-x}{dx}=\frac{y'-y}{dy}=\frac{z'-z}{dz};$$

and these are the equations to the tangent line.

Hence it appears, that the developable screw surface in Art. (218) is the envelop of the osculating planes of the helix.

(238) As an example of a developable surface considered as the envelop of a series of planes, take that generated by the normal planes to the spherical ellipse. The equation to a normal plane of this curve at the point x, y, z, is, by Art. (182),

$$a^2(b^2-c^2)\frac{x'}{x}+b^2(c^2-a^2)\frac{y'}{y}+c^2(a^2-b^2)\frac{z'}{z}=0\,\ldots\ldots(1),$$

x, y, z being connected by the equations

$$x^2 + y^2 + z^2 = r^2 \ldots\ldots(2), \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots(3).$$

Differentiating (1), (2) and (3) with respect to x, y, z, we have

$$a^2(b^2-c^2)\frac{x'}{x^2}\,dx + b^2(c^2-a^2)\frac{y'}{y^2}\,dy + c^2(a^2-b^2)\frac{z'}{z^2}\,dz \ldots(4),$$

$$xdx + ydy + zdz = 0 \ldots(5), \qquad \frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2} = 0 \ldots(6);$$

$(4) + \lambda(5) + \mu(6) = 0$ gives, on equating to zero the coefficients of each differential, the three equations

$$a^2(b^2-c^2)\frac{x'}{x^2} + \lambda x + \mu\frac{x}{a^2} = 0$$

$$b^2(c^2-a^2)\frac{y'}{y^2} + \lambda y + \mu\frac{y}{b^2} = 0$$

$$c^2(a^2-b^2)\frac{z'}{z^2} + \lambda z + \mu\frac{z}{c^2} = 0.$$

Multiply these by x, y, z respectively, and add; then, in virtue of (1), (2) and (3), we have

$$\lambda r^2 + \mu = 0,$$

so that the preceding equations may be written as

$$a^2(b^2-c^2)\frac{x'}{x^2} = \lambda\left(\frac{r^2}{a^2} - 1\right)x, \quad b^2(c^2-a^2)\frac{y'}{y^2} = \lambda\left(\frac{r^2}{b^2} - 1\right)y,$$

$$c^2(a^2-b^2)\frac{z'}{z^2} = \lambda\left(\frac{r^2}{c^2} - 1\right)z,$$

whence

$$x^3 = \frac{a^4}{\lambda}\frac{b^2-c^2}{r^2-a^2}x', \qquad y^3 = \frac{b^4}{\lambda}\frac{c^2-a^2}{r^2-b^2}y', \qquad z^3 = \frac{c^4}{\lambda}\frac{a^2-b^2}{r^2-c^2}z'.$$

But if we divide (2) by r^2, and subtract it from (3), we have

$$x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) = 0;$$

and if we substitute in this the preceding values of x, y, z, we may divide out $\lambda^{-\frac{2}{3}}$ and obtain an equation in x', y', z' which must evidently be of the form

$$\left(\frac{x'}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y'}{\beta}\right)^{\frac{2}{3}} + \left(\frac{z'}{\gamma}\right)^{\frac{2}{3}} = 0.$$

If this equation be cleared of fractional indices, it will be found to be a homogeneous function of the 6th degree equated to zero, and consequently the developable surface is in this case a cone; as indeed is otherwise obvious, since all the normal planes pass through the centre of the sphere.

(239) We showed in Art (195), that the generating line of a developable surface could have no more than two directors: these may be replaced by the conditions that the developable surface shall circumscribe two given surfaces, in which case the equation is to be found in the following manner. Let $x_1, y_1, z_1, x_2, y_2, z_2$ be the current co-ordinates of the two surfaces, of which the equations are

$$u_1 = 0, \qquad u_2 = 0 \ldots\ldots\ldots\ldots (1).$$

Then if the points $x_1, y_1, z_1, x_2, y_2, z_2$ lie on the same generator of the developable surface, the normals to the given surfaces at these points must be parallel to each other and perpendicular to the generator, since that line lies in a plane touching both surfaces. These conditions give us the relations

$$\left.\begin{array}{c} \dfrac{\frac{du_1}{dx_1}}{\frac{du_2}{dx_2}} = \dfrac{\frac{du_1}{dy_1}}{\frac{du_2}{dy_2}} = \dfrac{\frac{du_1}{dz_1}}{\frac{du_2}{dz_2}} \\ (x_1 - x_2)\dfrac{du_1}{dx_1} + (y_1 - y_2)\dfrac{du_1}{dy_1} + (z_1 - z_2)\dfrac{du_1}{dz_1} = 0 \end{array}\right\} \ldots . (2).$$

These three equations with the two equations to the surfaces, and the two equations to the generating line

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} \ldots\ldots\ldots (3),$$

give seven equations between which we may eliminate the six quantities $x_1, y_1, z_1, x_2, y_2, z_2$, and so find the equation to the envelop.

But it will generally be easier to proceed as follows: The equations (2) joined to $u_1 = 0$ enable us to eliminate x_2, y_2, z_2, and the result is the equation to a surface, which by its intersection with $u_1 = 0$ determines the curve of contact of the developable surface, and the surface $u_1 = 0$; the developable surface may then

be considered as the envelop of the tangent planes to that surface along the curve of contact.

(240) As an example, take the case of the two spheres

$$x_1^2 + y_1^2 + z_1^2 = r_1^2, \qquad (x_2 - \alpha)^2 + (y_2 - \beta)^2 + (z_2 - \gamma)^2 = r_2^2;$$

we deduce then from (2), the equations

$$\frac{x_1}{x_2 - \alpha} = \frac{y_1}{y_2 - \beta} = \frac{z_1}{z_2 - \gamma}, \qquad x_1x_2 + y_1y_2 + z_1z_2 = r_1^2;$$

from which we find

$$\frac{x_2 - \alpha}{x_1} = \frac{y_2 - \beta}{y_1} = \frac{z_2 - \gamma}{z_1} = \pm\frac{r_2}{r_1} = \frac{r_1^2 - (\alpha x_1 + \beta y_1 + \gamma z_1)}{r_1^2},$$

and therefore $\qquad \alpha x_1 + \beta y_1 + \gamma z_1 = r_1(r_1 \pm r_2) \,\ldots\ldots\ldots\, (a).$

This being the equation to two planes, shows that there are two circles which are curves of contact corresponding, one to an internal, the other to an external sheet. The equation to a tangent plane to the sphere is

$$xx_1 + yy_1 + zz_1 = r_1^2 \,\ldots\ldots\ldots\ldots\, (b),$$

x_1, y_1, z_1, being connected by the equation

$$x_1^2 + y_1^2 + z_1^2 = r_1^2 \,\ldots\ldots\ldots\ldots\, (c);$$

and by equation (a). We have now to find the locus of the ultimate intersections of the planes determined by equation (b).

Differentiating (b), (c) and (a), considering x_1, y_1, z_1, as variable, we have

$$xdx_1 + ydy_1 + zdz_1 = 0 \,\ldots\ldots\ldots\ldots\, (b')$$

$$x_1dx_1 + y_1dy_1 + z_1dz_1 = 0 \,\ldots\ldots\ldots\, (c'),$$

$$\alpha dx_1 + \beta dy_1 + \gamma dz_1 = 0 \,\ldots\ldots\ldots\, (a'),$$

$\lambda(b') = \mu(c') + (a')$ gives, on equating the coefficients of each differential,

$$\lambda x = \mu x_1 + \alpha, \qquad \lambda y = \mu y_1 + \beta, \qquad \lambda z = \mu z_1 + \gamma.$$

Multiplying these by x_1, y_1, z_1 respectively, and adding, we find, in virtue of (a) and (c),

$$\lambda r_1^2 = \mu r_1^2 + r_1(r_1 \pm r_2).$$

Substituting the value of μ derived from this equation in the first of the preceding, it becomes

$$\lambda r_1^2 (x - x_1) = r_1^2\alpha - r_1(r_1 \pm r_2)\,x_1;$$

and therefore, by the symmetry of the formulæ,

$$\frac{x - x_1}{r_1\alpha - (r_1 \pm r_2)\,x_1} = \frac{y - y_1}{r_1\beta - (r_1 \pm r_2)\,y_1} = \frac{z - z_1}{r_1\gamma - (r_1 \pm r_2)\,z_1}.$$

If we multiply numerator and denominator of each of these ratios by x, y, z respectively and add, each of them is by Art. (22) equal to

$$\frac{x^2 + y^2 + z^2 - r_1^2}{r_1(\alpha x + \beta y + \gamma z) - (r_1 \pm r_2)r_1^2}.$$

Again, if we do the same with α, β, γ, each is equal to

$$\frac{\alpha x + \beta y + \gamma z - r_1(r_1 \pm r_2)}{r_1(\alpha^2 + \beta^2 + \gamma^2) - r_1(r_1 \pm r_2)^2};$$

Equating these two expressions, we find

$$\{x^2+y^2+z^2 - r_1^2\}\{\alpha^2+\beta^2+\gamma^2 - (r_1 \pm r_2)^2\} = \{\alpha x + \beta y + \gamma z - r_1(r_1 \pm r_2)\}^2;$$

as the equation to the developable surface, which touches the two spheres. It is evident that it represents two cones, the vertex of one being between the two spheres, and of the other without.

(241) *Tubular Surfaces.* A tubular surface is the envelop of a sphere of constant radius, the centre of which moves along some line called the axis, either plane or of double curvature. Let the co-ordinates of the centre of the sphere be α, β, γ, then its equation may be written

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2 \ldots\ldots (1):$$

since the centre lies always in some given line, the two equations to this line give two relations between α, β, γ, so that as r is constant, there is only one independent parameter. By the general theory, the equation to the envelop will be found by eliminating α, β, γ between (1) and its differential

$$(x - \alpha)\,d\alpha + (y - \beta)\,d\beta + (z - \gamma)\,d\gamma = 0 \ldots\ldots (2),$$

combined with the two relations between α, β, γ.

The *characteristic* is determined by the combination of (1) and (2): and as (2) is evidently the equation to a plane, which passes through the point α, β, γ and is perpendicular to the axis, the characteristic is a great circle of the sphere; and the tubular surface may be supposed to be generated by the motion of a circle of constant radius, the centre of which moves along a curve to which its plane is always normal.

(242) Let the axis of the tubular surface be the straight line

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} \dots\dots\dots\dots\dots\dots (1);$$

then, since

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2 \dots\dots\dots (2),$$

$$(x - \alpha)\, d\alpha + (y - \beta)\, d\beta + (z - \gamma)\, d\gamma = 0 \dots\dots\dots (3).$$

But (1) gives us

$$\frac{d\alpha}{l} = \frac{d\beta}{m} = \frac{d\gamma}{n};$$

whence (3) becomes

$$l(x - \alpha) + m(y - \beta) + n(z - \gamma) = 0 \dots\dots (4).$$

Again from (1) we have

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{l\alpha + m\beta + n\gamma}{l^2 + m^2 + n^2} = lx + my + nz,$$

from (4), and therefore

$$\alpha = l(lx + my + nz), \quad \beta = m(lx + my + nz), \quad \gamma = n(lx + my + nz).$$

Substituting these values in (2), we have, as the equation to the envelop,

$$\{(1 - l^2)x - l(my + nz)\}^2 + \{(1 - m^2)y - m(lx + nz)\}^2 + \{(1 - n^2)z - n(lx + my)\}^2 = r^2,$$

which is that to a right circular cylinder having the line (1) as its axis.

(243) Let the axis of the tubular surface be the circle determined by the equations

$$\gamma = 0, \quad \alpha^2 + \beta^2 = c^2 \dots\dots\dots\dots (1);$$

then the equation to the sphere may be written as

$$(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2 \dots\dots\dots (2).$$

Since the characteristic is a great circle of the sphere (2), it appears that the envelop in this case is the same surface as that deduced in Art. (223), and it is unnecessary to repeat the analysis, but we may show how to find the *edge* of the envelop. The result of the elimination of $d\alpha$ and $d\beta$ between the differentials of (1) and (2) is

$$\beta x - \alpha y = 0 \dots\dots\dots\dots\dots (3),$$

the equation to a plane which, combined with (2), gives the characteristic. To determine the edge, we differentiate (3) with

respect to α and β, paying regard to (1), and after eliminating these quantities, we find

$$x^2 + y^2 = 0,$$

which, combined with the equation to the envelop, gives

$$z^2 = r^2 - c^2.$$

If $r > c$ this is possible, and combined with the preceding equation determines two points on the axis of z, which represent the edge of the envelop. If $r < c$ the equation is impossible, or there is no edge to the envelop, as, in fact, the consecutive characteristics do not meet.

The following are examples of envelops to surfaces, in which there are more parameters than one independent.

(244) Find the surface constantly touched by the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \dots\dots\dots\dots (1),$$

a, b, c being connected by the equation

$$abc = m^3 \dots\dots\dots\dots\dots\dots (2).$$

Differentiating (1) with respect to a, b, c, and taking the logarithmic differential of (2), we have

$$\frac{xda}{a^2} + \frac{ydb}{b^2} + \frac{zdc}{c^2} = 0 \dots .(3), \qquad \frac{da}{a} + \frac{db}{b} + \frac{dc}{c} = 0 \dots .(4),$$

$(3) + \lambda\,(4) = 0$ gives, on equating to zero the coefficient of each differential,

$$\frac{x}{a^2} = \frac{\lambda}{a}, \quad \frac{y}{b^2} = \frac{\lambda}{b}, \quad \frac{z}{c^2} = \frac{\lambda}{c}.$$

Multiplying by a, b, c respectively, and adding, we have, in virtue of (1),

$$\lambda = \tfrac{1}{3},$$

whence $\qquad a = 3x, \quad b = 3y, \quad c = 3z,$

which values, substituted in (3), give

$$xyz = \frac{m^3}{27}$$

as the equation to the envelop.

(245) Find the equation to the surface which is constantly touched by the plane

$$lx + my + nz = v,$$

$l, m, n, v,$ being connected by the equations

$$l^2 + m^2 + n^2 = 1,$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0.$$

Differentiating with respect to $l, m, n, v,$ we have

$$xdl + ydm + zdn = dv \ldots\ldots\ldots\ldots (1),$$

$$ldl + mdm + ndn = 0 \ldots\ldots\ldots\ldots (2),$$

$$\frac{ldl}{v^2 - a^2} + \frac{mdm}{v^2 - b^2} + \frac{ndn}{v^2 - c^2}$$

$$= vdv \left\{\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2}\right\} \ldots (3);$$

$\lambda\,(1) - \mu\,(2) - (3)$ gives, on equating the coefficients of each differential,

$$\lambda x = \mu l + \frac{l}{v^2 - a^2} \ldots\ldots\ldots\ldots (4),$$

$$\lambda y = \mu m + \frac{m}{v^2 - b^2} \ldots\ldots\ldots\ldots (5),$$

$$\lambda z = \mu n + \frac{n}{v^2 - c^2} \ldots\ldots\ldots\ldots (6),$$

$$\lambda = v \left\{\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2}\right\} \ldots\ldots (7);$$

$l\,(4) + m\,(5) + n\,(6)$ gives, by the conditions,

$$\lambda v = \mu \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$x(4) + y(5) + z(6)$ gives, putting $r^2 = x^2 + y^2 + z^2$,

$$\lambda r^2 = \mu v + \frac{lx}{v^2 - a^2} + \frac{my}{v^2 - b^2} + \frac{nz}{v^2 - c^2};$$

whence $$\lambda\,(r^2 - v^2) = \frac{lx}{v^2 - a^2} + \frac{my}{v^2 - b^2} + \frac{nz}{v^2 - c^2} \ldots\ldots (9)$$

$(4)^2 + (5)^2 + (6)^2$ gives

$$\lambda^2 r^2 = \mu^2 + \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2};$$

whence $\lambda^2 (r^2 - v^2) = \dfrac{\lambda}{v}$ by (7) and (8). ....(10),

and therefore $\lambda = \dfrac{1}{v\,(r^2 - v^2)}$, and $\mu = \dfrac{1}{r^2 - v^2}$.

Substituting these values in (4), we have

$$\frac{x}{v(r^2-v^2)} = l\left(\frac{1}{r^2-v^2} + \frac{1}{v^2-a^2}\right);$$

whence $$\frac{x}{r^2-a^2} = \frac{vl}{v^2-a^2}.$$

Similarly $$\frac{y}{r^2-b^2} = \frac{vm}{v^2-b^2};$$

and $$\frac{z}{r^2-c^2} = \frac{vn}{v^2-c^2}.$$

Multiply by x, y, z, and add, then by (9) and (10),

$$\frac{x^2}{r^2-a^2} + \frac{y^2}{r^2-b^2} + \frac{z^2}{r^2-c^2} = 1.$$

This is the equation to the surface of a wave of light propagated through a crystalline medium. See Fresnel, *Mémoires de l'Institut*, vol. VII. p. 136; Ampère, *Annales de Chimie et de Physique*, vol. XXXIX. p. 113; and Smith, *Cambridge Transactions*, vol. VI. p. 85.

CHAPTER XII.

ON PARTIAL DIFFERENTIAL EQUATIONS TO FAMILIES OF SURFACES.

(246) We have seen in the preceding Chapters that families of urfaces might be expressed by means of equations involving :bitrary functions, on the form of which depends any particular ıdividual surface of the family. We might eliminate by dif- ›rentiation the arbitrary functions from these equations, and ıus obtain partial differential equations to the families of urfaces: but it is equally possible to obtain these directly from ne equations to the generator, as we proceed to shew.

Let the equations to the generator be

$$f(x, y, z, a, b, c...) = 0, \quad f_1(x, y, z, a, b, c...) = 0...(1);$$

n which $a, b, c,$ ...are n parameters, connected by $n-1$ equa- ions of condition; so that there is only one really independent, f which the others may be considered as functions. Now, to egin, let there be only two parameters, a and b, of which b may e taken as a function of a: then, if we differentiate the two quations (1), first with regard to x, and next with regard to y, onsidering z, a, and b as functions of these variables, we obtain our additional equations, while we introduce three new quanti- ies, viz.

$$\frac{db}{da}, \frac{da}{dx}, \frac{da}{dy}.$$

We have therefore, on the whole, five quantities,

$$a, b, \frac{db}{da}, \frac{da}{dx}, \frac{da}{dy},$$

which may be eliminated between the six equations consisting of (1) and their four differentials. It is obvious that the result of the elimination must be a partial differential equation of the first order, since we proceed only to one differentiation.

If there be three parameters, a, b, c, on proceeding to the second differentials, we obtain twelve equations, but we have then to eliminate twelve quantities, viz.

$$a,\ b,\ c,\ \frac{db}{da},\ \frac{dc}{da},\ \frac{d^2b}{da^2},\ \frac{d^2c}{da^2},\ \frac{da}{dx},\ \frac{da}{dy},\ \frac{d^2a}{dx^2},\ \frac{d^2a}{dxdy},\ \frac{d^2a}{dy^2},$$

which is in general impossible; we must therefore proceed to the third differentiation when we find twenty equations between which we have to eliminate eighteen quantities, and the result gives two differential equations of the third order.

It is easy to find, in general, the order of differentiation to which we must proceed in order to eliminate m parameters. Let n be the required order of differentiation; then the number of quantities in the series

$$a,\ \frac{da}{dx},\ \frac{da}{dy},\ \frac{d^2a}{dx^2},\ \frac{d^2a}{dxdy},\ \frac{d^2a}{dy^2},\ldots\frac{d^na}{dx^n},\ldots\frac{d^na}{dy^n},$$

is $\frac{1}{2}(n+1)(n+2)$, while the successive differentials of the $m-1$ parameters $b, c\ldots$ with respect to a, together with the quantities themselves, give $(m-1)(n+1)$ functions; so that the total number of quantities to be eliminated is

$$\tfrac{1}{2}(n+1)(n+2)+(m-1)(n+1).$$

On the other hand, the number of equations, including the original ones, together with their differentials up to the n^{th} order inclusive, is $(n+1)(n+2)$.

In order, then, that elimination may be possible, we must have
$$(n+1)(n+2) > \tfrac{1}{2}(n+1)(n+2)+(m-1)(n+1),$$
$$\text{or}\quad \tfrac{1}{2}(n+2) > m-1,$$
from which the lowest value of n is
$$n = 2m-3.$$

(247) If the equations to the generator be given in the explicit form
$$u = c,\quad v = \phi(c)\ldots\ldots\ldots\ldots\ldots(1),$$
the partial differential equation to the family of surfaces is easily found. For, supposing the functional equation to be
$$F(x, y, z) = 0\ldots\ldots\ldots\ldots\ldots\ldots(2),$$
we have
$$\frac{dF}{dx}dx+\frac{dF}{dy}dy+\frac{dF}{dz}dz = 0\ldots\ldots\ldots\ldots(3).$$

Now, if the curve (1) lie on the surface (2), the values of the differentials dx, dy, dz, derived from (1), must satisfy equation (3). But if for shortness we put

$$\frac{du}{dy}\frac{dv}{dz} - \frac{du}{dz}\frac{dv}{dy} = P, \quad \frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz} = Q, \quad \frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx} = R,$$

we have from (1) $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots\dots\dots\dots\dots(4).$$

Eliminating dx, dy, dz, between (3) and (4), we find

$$P\frac{dF}{dx} + Q\frac{dF}{dy} + R\frac{dF}{dz} = 0,$$

as the required differential equation.

If we put $p = \dfrac{dz}{dx}$, $q = \dfrac{dz}{dy}$, we have

$$\frac{dF}{dx} + \frac{dF}{dz}p = 0, \quad \frac{dF}{dy} + \frac{dF}{dz}q = 0,$$

and therefore the partial differential equation may be written also under the form $$Pp + Qq = R.$$

Cylindrical Surfaces.

(248) The equations to the generator are, in this case,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

l, m, n, being constant: hence

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n}:$$

these equations, combined with

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0,$$

give us $$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0,$$

as the partial differential equation to cylindrical surfaces.

This equation may be applied to find the conditions that the general equation of the second degree may represent a cylinder. The form of the general equation is

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + E = 0,$$

and we deduce from the preceding equation

$$l(Ax+B'z+C'y+A'')+m(By+C'x+A'z+B'')+n(Cz+A'y+B'x+C'')=0.$$

Now, so long as the coefficients of x, y, z, in the latter of these two equations, are supposed to be finite, it is evident that it cannot hold good for all the values of the three variables which satisfy the former: we must have, then, since the coexistence of the two equations for all such values of the variables is required by the nature of the case,

$$lA + mC' + nB' = 0,$$
$$mB + nA' + lC' = 0,$$
$$nC + lB' + mA' = 0,$$
$$lA'' + mB'' + nC'' = 0.$$

These are four relations between only two independent quantities (since the variables are, in fact, any two of the ratios $l : m : n$); and therefore, in order that they may coexist, there must be two equations of condition between the constants. These are easily found by eliminating l, m, n, between the first three, and between the last and the first two, and the results are

$$AA'^2 + BB'^2 + CC'^2 - ABC - 2A'B'C' = 0,$$
$$A''(A'C' - BB') + B''(B'C' - AA') + C''(AB - C'^2) = 0.$$

Conical Surfaces.

(249) The equations to the generator may be written

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

where l, m, n, are the parameters, and a, β, γ, constant.

From these we have

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n};$$

and, dividing each member of the latter equations by the corresponding one of the former,

$$\frac{dx}{x-a} = \frac{dy}{y-\beta} = \frac{dz}{z-\gamma};$$

by means of which equations, eliminating the differentials from

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0,$$

we find $$(x-\alpha)\frac{dF}{dx}+(y-\beta)\frac{dF}{dy}+(z-\gamma)\frac{dF}{dz}=0,$$

as the differential equation to conical surfaces.

If we make $\alpha=0$, $\beta=0$, $\gamma=0$, that is, if we suppose the vertex of the cone to be at the origin, the equation becomes

$$x\frac{dF}{dx}+y\frac{dF}{dy}+z\frac{dF}{dz}=0,$$

shewing that F is a homogeneous function of x, y, z.

Conoidal Surfaces.

(250) If the axis of the surface be the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

the equations to the generator will be

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}\ \ldots\ldots\ldots\ldots(1),$$

α', β', γ', being the point of the axis through which the generator passes, the parameters of the equations being subject to the relations $$ll'+mm'+nn'=0\ \ldots\ldots\ldots\ldots(2),$$

$$\frac{\alpha'-\alpha}{l}=\frac{\beta'-\beta}{m}=\frac{\gamma'-\gamma}{n}\ \ldots\ldots\ldots\ldots(3).$$

From (1) there is $$\frac{dx}{l'}=\frac{dy}{m'}=\frac{dz}{n'}\ \ldots\ldots\ldots\ldots(4).$$

But, from (1), (2), and (3), we have, representing each of the members of (3) by r,

$$l(x-\alpha-lr)+m(y-\beta-mr)+n(z-\gamma-nr)=0\,;$$

whence, observing that $l^2+m^2+n^2=1$, we have

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=r\ \ldots\ldots(5).$$

From (1) and (4), we get

$$\frac{dx}{x-\alpha'}=\frac{dy}{y-\beta'}=\frac{dz}{z-\gamma'}\,;$$

hence, from (3),

$$\frac{dx}{x-\alpha-lr}=\frac{dy}{y-\beta-mr}=\frac{dz}{z-\gamma-nr}\ \ldots\ldots(6).$$

But supposing $F = 0$ to be the equation to the surface, we have

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0\,;$$

and therefore, by virtue of (6), the differential equation to conoidal surfaces will be

$$(x - a - lr)\frac{dF}{dx} + (y - \beta - mr)\frac{dF}{dy} + (z - \gamma - nr)\frac{dF}{dz} = 0,$$

when, for r, we substitute the expression given in (5).

If the axis of the conoid be taken as the axis of z, then $l = 0$, $m = 0$, $n = 1$, $a = 0$, $\beta = 0$, and therefore $z - \gamma - nr = 0$; the differential equation will therefore become

$$x\frac{dF}{dx} + y\frac{dF}{dy} = 0.$$

Surfaces of Revolution.

(251) If the equations to the axis be

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

the equations to the generator will be

$$lx + my + nz = c, \quad (x-a)^2 + (y-\beta)^2 + (z-\gamma)^2 = c_1\,;$$

whence
$$ldx + mdy + ndz = 0,$$

$$(x-a)\,dx + (y-\beta)\,dy + (z-\gamma)\,dz = 0,$$

and therefore

$$\frac{dx}{m(z-\gamma) - n(y-\beta)} = \frac{dy}{n(x-a) - l(z-\gamma)} = \frac{dz}{l(y-\beta) - m(x-a)}:$$

which equations, combined with

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0,$$

give
$$\{m(z-\gamma) - n(y-\beta)\}\frac{dF}{dx} + \{n(x-a) - l(z-\gamma)\}\frac{dF}{dy}$$
$$+ \{l(y-\beta) - m(x-a)\}\frac{dF}{dz} = 0,$$

as the differential equation to surfaces of revolution.

If the axis of revolution be taken as the axis of z, then $l = 0$, $m = 0$, $n = 1$, $a = 0$, $\beta = 0$, so that the differential equation becomes

$$y\frac{dF}{dx} - x\frac{dF}{dy} = 0.$$

Ruled Surfaces having a Director Plane.

(252) If we take the director-plane as that of (x, y), the equations to the generator are

$$z = a, \quad y = x\phi(a) + \Psi(a).$$

From the first, $$p = \frac{da}{dx}, \quad q = \frac{da}{dy}.$$

From the second,

$$0 = \phi(a) + \{x\phi'(a) + \Psi'(a)\} p, \quad 1 = \{x\phi'(a) + \Psi'(a)\} q,$$

whence $$\frac{p}{q} = -\phi(a).$$

Differentiating this equation first with respect to x, and then with respect to y, and putting

$$\frac{d^2z}{dx^2} = r, \quad \frac{d^2z}{dxdy} = s, \quad \frac{d^2z}{dy^2} = t,$$

we find $$\frac{qr - ps}{q^2} = -\phi'(a).p, \quad \frac{qs - pt}{q^2} = -\phi'(a).q,$$

whence $$q^2r - 2pqs + p^2t = 0,$$

which is the required differential equation.

Developable Surfaces.

(253) The equations to this family are, by Art. (235),

$$ax + by + cz = 1 \ldots\ldots\ldots\ldots (1),$$
$$xda + ydb + zdc = 0 \ldots\ldots\ldots\ldots (2).$$

Differentiating (1) first with respect to x, and next with respect to y, considering a, b, c, z, as all variable, we have, in virtue of (2), $$a + pc = 0, \quad b + qc = 0.$$

Again, differentiating these equations with respect both to x and y, we obtain the four equations,

$$cr + \frac{da}{dx} + p\frac{dc}{dx} = 0, \quad cs + \frac{da}{dy} + p\frac{dc}{dy} = 0,$$

$$ct + \frac{db}{dy} + q\frac{dc}{dy} = 0, \quad cs + \frac{db}{dx} + q\frac{dc}{dx} = 0;$$

from which we easily deduce

$$rt - s^2 = 0,$$

or
$$\frac{d^2z}{dx^2}\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dxdy}\right)^2=0,$$
as the differential equation to developable surfaces.

Tubular Surfaces.

(254) If the axis be a plane curve, the equations to this family of surfaces are
$$(x-a)^2+(y-b)^2+z^2=\rho^2 \dots\dots\dots\dots (1),$$
$$(x-a)\,da+(y-b)\,db=0 \dots\dots\dots\dots (2).$$
Differentiating (1) first with regard to x, and next with regard to y, considering a, b, and z all variable, we have, in virtue of (2),
$$x-a+pz=0, \quad y-b+qz=0.$$
Substituting in (1) the values of a and b, given by these equations, we have
$$z^2(1+p^2+q^2)=\rho^2,$$
as the required differential equation.

If the axis be not a plane curve, then the equations to this family are
$$(x-a)^2+(y-b)^2+(z-c)^2=\rho^2 \dots\dots\dots (1),$$
$$(x-a)\,da+(y-b)\,db+(z-c)\,dc=0 \dots\dots\dots (2).$$
Differentiating (1) as before, first with respect to x, and next with respect to y, we have, in virtue of (2),
$$x-a+p(z-c)=0, \quad y-b+q(z-c)=0 \dots (3).$$
Differentiating these in like manner, we have
$$\{1+p^2+r(z-c)\}\,dx-(da+pdc)=0,$$
$$\{pq+s(z-c)\}\,dy-(da+pdc)=0,$$
$$\{1+q^2+t(z-c)\}\,dy-(db+qdc)=0,$$
$$\{pq+s(z-c)\}\,dx-(db+qdc)=0.$$
From these four equations we easily find
$$\{1+p^2+r(z-c)\}\{1+q^2+t(z-c)\}-\{pq+s(z-c)\}^2=0.$$
But, on combining (1) and (3), we have
$$(1+p^2+q^2)(z-c)^2=\rho^2,$$
whence
$$z-c=\pm\frac{\rho}{(1+p^2+q^2)^{\frac{1}{2}}},$$
which value, substituted in the preceding equation, gives, after obvious reduction,
$$(1+p^2+q^2)^2\pm\rho\,(1+p^2+q^2)^{\frac{1}{2}}.\{r(1+q^2)-2pqs+t(1+p^2)\}+(rt-s^2)\,\rho^2=0,$$
as the required differential equation.

The partial differential equations of developable and tubular surfaces may be investigated also symmetrically in the following manner.

Developable Surfaces.

(255) Let x, y, z, be the co-ordinates of any point of a developable surface; α, β, γ, the variable parameters. Then

$$\alpha x + \beta y + \gamma z = 1 \ldots\ldots\ldots\ldots\ldots (1),$$

$$x d\alpha + y d\beta + z d\gamma = 0 \ldots\ldots\ldots\ldots\ldots (2);$$

α, β, γ, being subject to two equations,

$$\left.\begin{array}{l} \phi(\alpha, \beta, \gamma) = 0 \\ \chi(\alpha, \beta, \gamma) = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (3),$$

the functions ϕ and χ being any whatever.

From (1) and (2) there is also

$$\alpha dx + \beta dy + \gamma dz = 0 \ldots\ldots\ldots\ldots (4).$$

Suppose $F = 0$ to be the equation to the developable surface; and put

$$\frac{dF}{dx} = U, \quad \frac{dF}{dy} = V, \quad \frac{dF}{dz} = W,$$

$$\frac{d^2F}{dx^2} = u, \quad \frac{d^2F}{dy^2} = v, \quad \frac{d^2F}{dz^2} = w,$$

$$\frac{d^2F}{dydz} = u', \quad \frac{d^2F}{dzdx} = v', \quad \frac{d^2F}{dxdy} = w':$$

then we shall also have

$$Udx + Vdy + Wdz = 0 \ldots\ldots\ldots\ldots (5).$$

By the aid of an indeterminate multiplier, λ, we shall get from (4) and (5), observing that, by virtue of (1), (2), (3), x and y may be regarded as independent variables,

$$\left.\begin{array}{l} \alpha = \dfrac{U}{\lambda} \\ \beta = \dfrac{V}{\lambda} \\ \gamma = \dfrac{W}{\lambda} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now the only equations connecting $\alpha, \beta, \gamma, x, y, z$, with $d\alpha, d\beta, d\gamma$, are (2), and the differentials of (3); all which three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of $x, y, z, \alpha, \beta, \gamma$. Differentiating, then, the equations (6) on this hypothesis, we get

$$\frac{d\lambda}{\lambda} = \frac{dU}{U} = \frac{1}{U} \cdot (udx + w'dy + v'dz),$$

$$\frac{d\lambda}{\lambda} = \frac{dV}{V} = \frac{1}{V} \cdot (vdy + u'dz + w'dx),$$

$$\frac{d\lambda}{\lambda} = \frac{dW}{W} = \frac{1}{W} \cdot (wdz + v'dx + u'dy).$$

Eliminating dy and dz by cross-multiplication, we get

$$\frac{d\lambda}{\lambda} \cdot \{U(vw - u'^2) + V(u'v' - ww') + W(w'u' - vv')\} = Rdx \ldots (7),$$

where $\qquad R = uvw - uu'^2 - vv'^2 - ww'^2 + 2u'v'w'.$

Observing that R is a symmetrical function of u, v, w, u', v', w', it is evident that we shall have also

$$\frac{d\lambda}{\lambda} \cdot \{V(wu - v'^2) + W(v'w' - uu') + U(u'v' - ww')\} = Rdy \ldots (8),$$

$$\frac{d\lambda}{\lambda} \cdot \{W(uv - w'^2) + U(w'u' - vv') + V(v'w' - uu')\} = Rdz \ldots (9).$$

Multiplying the equations (7), (8), (9), by U, V, W, respectively, adding, and attending to (5), we get

$$U^2(vw - u'^2) + V^2(wu - v'^2) + W^2(uv - w'^2)$$
$$+ 2VW(v'w' - uu') + 2WU(w'u' - vv') + 2UV(u'v' - ww') = 0,$$

as the symmetrical form of the partial differential equation of developable surfaces.

Tubular Surfaces.

(256) Let ρ be the radius of each sphere; α, β, γ, the co-ordinates of the centre of any one of the spheres; then, x, y, z, being the co-ordinates of any point of the envelop, we shall have

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \rho^2 \ldots\ldots\ldots (1),$$

$$(x - \alpha)\, d\alpha + (y - \beta)\, d\beta + (z - \gamma)\, d\gamma = 0 \ldots\ldots\ldots (2).$$

The quantities α, β, γ, are subject to two equations,

$$\left.\begin{array}{l} f(\alpha, \beta, \gamma) = 0 \\ \phi(\alpha, \beta, \gamma) = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2), we get

$$(x-\alpha)\,dx + (y-\beta)\,dy + (z-\gamma)\,dz = 0 \;\ldots\ldots(4).$$

Suppose $F = 0$ to be the equation to the tubular surface; then we shall have also $\quad Udx + Vdy + Wdz = 0 \;\ldots\ldots\ldots\ldots(5).$

Now $\alpha, \beta, \gamma, x, y, z$, being connected by the equations (1), (2), (3), it is evident that α, β, γ, z, may be regarded as functions of two independent variables, x and y: we have then, from (4) and (5), by the aid of an indeterminate multiplier λ,

$$\left.\begin{aligned} \lambda U + x - \alpha &= 0 \\ \lambda V + y - \beta &= 0 \\ \lambda W + z - \gamma &= 0 \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots (6).$$

Now the only equations connecting $\alpha, \beta, \gamma, x, y, z$, with $d\alpha$, $d\beta$, $d\gamma$, are (2), and the differentials of (3); but all these three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of $x, y, z, \alpha, \beta, \gamma$: differentiating, then, the equations (6) on this hypothesis, we get

$$-\,d\lambda = \frac{\lambda dU + dx}{U} = \frac{\lambda dV + dy}{V} = \frac{\lambda dW + dz}{W},$$

and therefore, performing the differentiations,

$$\begin{aligned} (1+\lambda u)\,dx + \lambda w'dy + \lambda v'dz &= -\,Ud\lambda, \\ (1+\lambda v)\,dy + \lambda u'dz + \lambda w'dx &= -\,Vd\lambda, \\ (1+\lambda w)\,dz + \lambda v'dx + \lambda u'dy &= -\,Wd\lambda. \end{aligned}$$

Eliminating dy and dz from these three equations, we get

$$\begin{aligned} -\frac{Rdx}{d\lambda} &= U + \lambda\left\{U(v+w) - Vw' - Wv'\right\} \\ &\quad + \lambda^2\left\{U(vw - u'^2) + u'(Vv' + Ww') - Vww' - Wvv'\right\} \ldots (7), \end{aligned}$$

where

$$R = (1+\lambda u)(1+\lambda v)(1+\lambda w) - \lambda^2(u'^2 + v'^2 + w'^2) - \lambda^3(uu'^2 + vv'^2 + ww'^2 - 2u'v'w'),$$

a symmetrical function of u, v, w, u', v', w'. We must have, therefore, also

$$\begin{aligned} -\frac{Rdy}{d\lambda} &= V + \lambda\left\{V(w+u) - Wu' - Uw'\right\} \\ &\quad + \lambda^2\left\{V(wu - v'^2) + v'(Ww' + Uu') - Wuu' - Uww'\right\} \ldots (8), \end{aligned}$$

$$-\frac{Rdz}{d\lambda} = W + \lambda\{W(u+v) - Uv' - Vu'\}$$
$$+ \lambda^2\{W(uv - w'^2) + w'(Uu' + Vv') - Uvv' - Vuu'\} \ldots (9).$$

Multiplying the equations (7), (8), (9), by U, V, W, respectively, adding, and paying attention to (5), we get

$$0 = U^2 + V^2 + W^2 + \lambda\{u(V^2+W^2) + v(W^2+U^2) + w(U^2+V^2)$$
$$- 2u'VW - 2v'WU - 2w'UV\}$$
$$+ \lambda^2\{U^2(vw - u'^2) + V^2(wu - v'^2) + W^2(uv - w'^2)$$
$$+ 2VW(v'w' - uu') + 2WU(w'u' - vv') + 2UV(u'v' - ww')\} = 0:$$

but, from (1) and (6), $\lambda^2 = \dfrac{\rho^2}{U^2 + V^2 + W^2}$;

hence we obtain, for the symmetrical form of the differential equation to tubular surfaces,

$$(U^2+V^2+W^2)^2 \pm \rho(U^2+V^2+W^2)^{\frac{3}{2}}.\{u(V^2+W^2) + v(W^2+U^2)$$
$$+ w(U^2+V^2) - 2u'VW - 2v'WU - 2w'UV\}$$
$$+ \rho^2\{U^2(vw - u'^2) + V^2(wu - v'^2) + W^2(uv - w'^2)$$
$$+ 2VW(v'w' - uu') + 2WU(w'u' - vv') + 2UV(u'v' - ww')\} = 0.*$$

(257) The transformation of the partial differential equations from the symmetrical to the unsymmetrical form is readily effected. Suppose, in fact, the equation $F = 0$ to be reduced to the form
$$F = z - f(x, y) = 0:$$
then it is easily seen that, p, q, r, s, t, denoting the partial differential coefficients of z with respect to x and y, according to the usual notation,

$$U = -p, \quad V = -q, \quad W = 1,$$
$$u = -r, \quad v = -t, \quad w = 0,$$
$$u' = 0, \quad v' = 0, \quad w' = -s.$$

If we substitute these values of the partial differential coefficients of F in the partial differential equation to the surface, we shall

* The symmetrical investigations of the differential equations of Developable and Tubular Surfaces, given in the text, have been extracted from the *Cambridge Mathematical Journal*, for November, 1844.

at once effect the proposed transformations. Thus, the equation to developable surfaces becomes

$$rt - s^2 = 0,$$

and the equation to tubular surfaces

$$(1 + p^2 + q^2)^2 \pm \rho (1 + p^2 + q^2)^{\frac{1}{2}} \{r (1 + q^2) - 2pqs + t (1 + p^2)\} + \rho^2 (rt - s^2) = 0,$$

the equations to these surfaces obtained in Arts. (253) and (254).

CHAPTER XIII.

ON SINGULAR POINTS AND LINES OF SURFACES.

(258) In the Chapter on Tangencies it was stated that, under certain circumstances, the equation to the tangent plane becomes nugatory in consequence of the vanishing of all the terms. We now proceed to consider the nature of the points where this occurs. It is to be observed that, since the vanishing of the three differential coefficients $\frac{dF}{dx}$, $\frac{dF}{dy}$, $\frac{dF}{dz}$, involves either one or two relations between x, y, z, besides the equation to the surface, the singularity can occur only at isolated points or along isolated lines, and not throughout any extent of surface.

In this and succeeding investigations we shall have frequent occasion to use the differential coefficients of the first and second order of a function of three variables: we shall therefore, for shortness, use the following notation. If the equation to the surface, cleared of radicals and fractions, be expressed by the equation

$$F(x, y, z) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

then we shall put

$$\frac{dF}{dx} = U, \quad \frac{dF}{dy} = V, \quad \frac{dF}{dz} = W,$$

$$\frac{d^2F}{dx^2} = u, \quad \frac{d^2F}{dy^2} = v, \quad \frac{d^2F}{dz^2} = w,$$

$$\frac{d^2F}{dydz} = u', \quad \frac{d^2F}{dzdx} = v', \quad \frac{d^2F}{dxdy} = w'.$$

Now at any point (x, y, z), let the variables receive the incre-

ments dx, dy, dz, then, by Taylor's Theorem, the equation (1) becomes

$$F(x, y, z) + Udx + Vdy + Wdz + \tfrac{1}{2}(udx^2 + vdy^2 + wdz^2) + u'dydz + v'dzdx + w'dxdy + \&c. = 0 \quad \dots\dots (2).$$

The conditions for a singular point are

$$U = 0, \quad V = 0, \quad W = 0 \quad \dots\dots (3);$$

which, together with (1), reduce equation (2) to

$$(u)\,dx^2 + (v)\,dy^2 + (w)\,dz^2 + 2\{(u')\,dydz + (v')\,dzdx + (w')\,dxdy\} + \&c. = 0 \dots (4),$$

the bracketed letters indicating the values which they take when we substitute for x, y, z, their values at the point in question.

If we suppose dx, dy, dz, to be the limits of the increments of the variables, the equation (4) will, at the limit, be reduced to the terms involving the lowest powers of these quantities; that is, to

$$(u)dx^2+(v)dy^2+(w)dz^2+2\{(u')dydz+(v')dzdx+(w')dxdy\} = 0 \dots (5).$$

If all the quantities (u), (v), (w), (u'), (v'), (w'), be zero, we must retain the terms of the development which involve partial differential coefficients of F of the third order, neglecting the remaining terms of the series, and so on successively, until we arrive at an order of partial differential coefficients of which, at any rate, all do not vanish. In the examples which we shall adduce, however, we shall not have occasion to proceed beyond second differential coefficients.

Equation (5) gives a relation subsisting between the increments dx, dy, dz, in the surface at the singular point. These are the same for the surface and for a straight line touching it at the point; and therefore equation (5) gives a relation between the increments dx, dy, dz, on the tangent lines at the singular points: or, since the very same relation must hold good also for the co-ordinates of all points of these lines, we may substitute x, y, z, for dx, dy, dz, in (5), and we find

$$(u)\,x^2 + (v)\,y^2 + (w)\,z^2 + 2\,(u')\,yz + 2\,(v')\,zx + 2\,(w')\,xy = 0 \dots (6)$$

as the equation to the locus of the tangent lines at the singular point, which is taken as the origin of co-ordinates. This equa-

tion, except for particular values of the coefficients, is that of a cone of the second degree. It may happen that this equation may be decomposed into two factors of the first degree, and then it will represent two planes. The condition that this may be the case is

$$(u)(v)(w) - (u)(u')^2 - (v)(v')^2 - (w)(w')^2 + 2(u')(v')(w') = 0.$$

If it had been necessary to proceed to third differential coefficients, we should have found generally the equation to a cone of the third degree, and so on: exceptions arising from the same cause as in the instance of the equation of second differential coefficients.

From the nature of the singular points which we have been investigating, it is evident that more than one tangent plane will belong to them; an infinite number in the case of the tangent cone, which may be regarded as the locus of the ultimate intersections of the tangent planes at the point. It appears therefore, as might have been anticipated, that a plurality of tangent planes is indicated by the indeterminate forms assumed by the direction-cosines of tangency at the point.

If the three equations (3) are satisfied by assigning certain relations between the variables, then the curve formed by the intersection of the surface (1) with that indicated by the relation between the variables which satisfies (3), is a locus of singular points; that is to say, it is a line in which two or more sheets of the surface intersect, at each point of which line the surface will of course have two or more tangent planes. If the equations (1) and (3) are satisfied simultaneously by assigning certain definite values to x, y, z, and not when they receive values differing slightly from these, the point will be single and not one of a series of singular points, and will have a tangent cone. If for possible values of two of the variables on one side of the singular point we find impossible values of the third variable, that point is a *cusp*. If the same occur at every point of the singular line, it is called an *edge of regression* (*arète de rebroussement*): such, for examples, are the curves which are the loci of the ultimate intersections of the generating lines of developable surfaces.

Ex. 1. Find the nature of the point at the origin in the surface
$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2-c^2z^2.$$

Here, putting $x^2+y^2+z^2=r^2$,
$$U = 2x(2r^2-a^2),$$
$$V = 2y(2r^2-b^2),$$
$$W = 2z(2r^2+c^2),$$
$$u = 2(2r^2-a^2)+8x^2,$$
$$v = 2(2r^2-b^2)+8y^2,$$
$$w = 2(2r^2+c^2)+8z^2,$$
$$u' = 8yz,\quad v' = 8zx,\quad w' = 8xy.$$

Now when $x=0$, $y=0$, $z=0$, U, V, W, all vanish, while
$$u=-2a^2,\quad v=-2b^2,\quad w=2c^2,\quad u'=0,\quad v'=0,\quad w'=0,$$
so that the equation to the tangent cone at the origin is
$$a^2x^2+b^2y^2-c^2z^2=0.$$

Ex. 2. The equation to Fresnel's wave-surface in biaxal crystals is
$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2$$
$$-c^2(a^2+b^2)z^2+a^2b^2c^2=0\,;$$
find whether it has singular points, and determine their nature.

Here $\quad U = 2x\left\{a^2(r^2-b^2-c^2)+a^2x^2+b^2y^2+c^2z^2\right\},$
$$V = 2y\left\{b^2(r^2-c^2-a^2)+a^2x^2+b^2y^2+c^2z^2\right\},$$
$$W = 2z\left\{c^2(r^2-a^2-b^2)+a^2x^2+b^2y^2+c^2z^2\right\},$$
where
$$r^2=x^2+y^2+z^2.$$
Now if we put $y=0$, $r^2=b^2$, and assume accordingly
$$x=\pm c\left(\frac{a^2-b^2}{a^2-c^2}\right)^{\frac{1}{2}},\quad z=\pm a\left(\frac{b^2-c^2}{a^2-c^2}\right)^{\frac{1}{2}},$$
we shall satisfy the equation to the surface, and also make U, V, and W vanish: hence, as the double signs of x and z may be combined in four different ways, there are four singular points on the surface. To obtain the equation to the tangent cone we must find the values of u, v, w, u', v', w', at the singular points. These are readily seen to be
$$u=8a^2c^2.\frac{a^2-b^2}{a^2-c^2},\quad v=-2(a^2-b^2)(b^2-c^2),\quad w=8a^2c^2\frac{b^2-c^2}{a^2-c^2},$$
$$u'=0,\quad v'=4ac\left\{(a^2-b^2)(b^2-c^2)\right\}^{\frac{1}{2}}\frac{a^2+c^2}{a^2-c^2},\quad w'=0.$$

Substituting these values, and dividing the whole by

$$8a^2c^2\frac{(a^2-b^2)(b^2-c^2)}{a^2-c^2},$$

we find, as the equation to the cone,

$$\frac{x^2}{b^2-c^2}-\frac{a^2-c^2}{4a^2c^2}y^2+\frac{z^2}{a^2-b^2}+\frac{a^2+c^2}{\{(a^2-b^2)(b^2-c^2)\}^{\frac{1}{2}}}\cdot\frac{xz}{ac}=0.$$

The existence of these singular points in the wave-surface was first pointed out by Sir W. Hamilton.

Ex. 3. Let the equation to the surface be

$$z(x^2+y^2+z^2)+ax^2+by^2=0;$$

then $\quad U=2x(z+a),\quad V=2y(z+b),\quad W=x^2+y^2+3z^2.$

At the origin, where $x=0,\ y=0,\ z=0$, these three quantities vanish; therefore there is a singular point at the origin: also

$$u=2(z+a),\quad v=2(z+b),\quad w=6z,$$

$$u'=2y,\qquad v'=2x,\qquad w'=0,$$

$$(u)=2a,\quad (v)=2b,\quad (w)=0,\quad (u')=0,\quad (v')=0,\quad (w')=0.$$

The equation to the locus of the tangent lines becomes, then,

$$ax^2+by^2=0,$$

which, a and b being supposed to be both positive, can only represent the axis of z. The cone in this case degenerates into a straight line; and, as z can never be positive, since that renders x and y impossible, it appears that the point under consideration is a cusp. The surface surrounds the negative axis of z, which it touches at the origin, so that its form resembles the shape of the flower of the convolvulus.

If a and b be of contrary signs, the equation to the locus of the tangent lines is $\qquad ax^2-by^2=0,$

which represents two planes perpendicular to the plane of x, y.

Ex. 4. Let the surface be the cono-cuneus of Wallis, the equation to which is $\ a^2y^2-x^2(c^2-z^2)=0.$

Here $\qquad U=-2x(c^2-z^2),\quad V=2a^2y,\quad W=2x^2z.$

These all vanish when $x=0,\ y=0$, independently of the value

of z; hence the axis of z is a locus of singular points or a singular line.

$$u = -2(c^2 - z^2), \quad v = 2a^2, \quad w = 2x^2,$$
$$u' = 0, \qquad v' = 4xz, \quad w' = 0.$$

The equation to the tangent lines becomes, in this case,

$$a^2 y'^2 - (c^2 - z^2) x'^2 = 0,$$

where x', y', are accentuated to distinguish them from z, the undetermined co-ordinate of the point of contact. The preceding equation is equivalent to those of two planes perpendicular to the plane of xy,

$$ay' + (c^2 - z^2)^{\frac{1}{2}} x' = 0,$$
$$ay' - (c^2 - z^2)^{\frac{1}{2}} x' = 0.$$

By assigning different values to z, we obtain different equations corresponding to successive points taken along the axis of z.

Ex. 5. The equation to the *héliçoide dévelopable* is

$$x \sin\left\{\frac{2\pi z}{h} - \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} + y \cos\left\{\frac{2\pi z}{h} - \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} = a.$$

Putting $\dfrac{2\pi z}{h} - \dfrac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a} = \theta$, it may be ascertained that

$$U = \sin\theta - \frac{x(x\cos\theta - y\sin\theta)}{a(x^2 + y^2 - a^2)^{\frac{1}{2}}};$$

$$V = \cos\theta - \frac{y(x\cos\theta - y\sin\theta)}{a(x^2 + y^2 - a^2)^{\frac{1}{2}}};$$

$$W = \frac{2\pi}{h}(x\cos\theta - y\sin\theta).$$

But, as may easily be shewn,

$$x\cos\theta - y\sin\theta = (x^2 + y^2 - a^2)^{\frac{1}{2}};$$

therefore, if we assume

$$x = a\sin\frac{2\pi z}{h}, \quad y = a\cos\frac{2\pi z}{h},$$

the preceding expressions will vanish, and therefore the line determined by these equations, and the equation to the surface, is a *locus* of singular points.

This line is the intersection of the surface by the cylinder

$$x^2 + y^2 = a^2,$$

and is evidently the generating helix. Since in the equation to the surface $x^2 + y^2$ can never be less than a^2, it appears that no part of the surface lies within the helix, which is therefore truly an edge of regression.

On proceeding to the second differential coefficients, and substituting in them the critical values of x and y, we find, retaining only the terms which become infinite from involving $(x^2+y^2-a^2)^{\frac{1}{2}}$ in the denominator,

$$(u) = -2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (v) = 2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (w) = 0,$$

$$(u') = \frac{2\pi}{h}\,a\cos\frac{2\pi z}{h}, \quad (v') = \frac{2\pi}{h}\,a\sin\frac{2\pi z}{h},$$

$$(w') = \sin^2\frac{2\pi z}{h} - \cos^2\frac{2\pi z}{h};$$

so that the equation to the locus of the tangent lines is

$$(y'^2 - x'^2)\,xy + x'y'\,(x^2 - y^2) + 2\pi\frac{a^2}{h}\,z'\,(x'x + y'y) = 0,$$

where the accentuated letters are the current co-ordinates of the tangents, and the unaccentuated the undetermined co-ordinates of the point of contact. This equation may be decomposed into two factors

$$y'x - x'y + 2\pi\frac{a^2}{h}\,z' = 0,$$

$$x'x + y'y = 0,$$

which are the equations to two planes.

(259) *There is a species of singular lines on certain surfaces of an entirely different character from the singular lines of which we have treated above. It occasionally happens that a single tangent plane will touch a surface not only at one point, but in a series of points forming a curve line. We shall investigate the analytical condition for the existence of such singular lines in surfaces.

The equation to the tangent-plane at any point x, y, z, is

$$Ux' + Vy' + Wz' = Ux + Vy + Wz.$$

* Singular lines of this species were first discussed generally by Mr. Greatheed, of Trinity College, in the *Cambridge Mathematical Journal*, vol. II. p. 22.

Let the right-hand member of this equation be represented by P: then, if the plane touches the surface in a curve line, $\frac{U}{P}, \frac{V}{P}, \frac{W}{P}$, remain constant while the co-ordinates vary in agreement with the condition

$$F(x, y, z) = 0,$$

and to another condition, which, together with that, determines the curve.

Then, as $\frac{U}{P}, \frac{V}{P}, \frac{W}{P}$, are all constant,

$$\frac{dU}{U} = \frac{dV}{V} = \frac{dW}{W} = \frac{dP}{P}:$$

hence, effecting the differentiations, and denoting each member of this multiple equation by dQ, we get

$$dU = udx + w'dy + v'dz = UdQ,$$
$$dV = vdy + u'dz + w'dx = VdQ,$$
$$dW = wdz + v'dx + u'dy = WdQ.$$

Eliminating dy and dz by cross-multiplication,

$$Rdx = \{U(vw - u'^2) + V(u'v' - ww') + W(w'u' - vv')\}\, dQ,$$

where $R = uvw - (uu'^2 + vv'^2 + ww'^2) + 2u'v'w'$,

a symmetrical function of u, v, w, u', v', w'. Similarly we must have

$$Rdy = \{V(wu - v'^2) + W(v'w' - uu') + U(u'v' - ww')\}\, dQ,$$
$$Rdz = \{W(uv - w'^2) + U(w'u' - vv') + V(v'w' - uu')\}\, dQ.$$

But, from the equation to the surface, we have also the condition

$$Udx + Vdy + Wdz = 0.$$

Hence, multiplying the previous equations by U, V, W, respectively, and adding, the first side of the equation disappears by the last condition, and we have

$$U^2(vw - u'^2) + V^2(wu - v'^2) + W^2(uv - w'^2)$$
$$+ 2VW(v'w' - uu') + 2WU(w'u' - vv') + 2UV(u'v' - ww') = 0;$$

which equation, combined with $F(x, y, z) = 0$, determines the curve of contact; and this condition must subsist, in order that the surface may be touched by the tangent plane in a curve.

This condition may be expressed more briefly in terms of the partial differential coefficients of z taken with respect to x and y. Conceive the equation to the surface to be put under the form

$$f(x, y) - z = 0:$$

then $$U = p, \quad V = q, \quad W = -1, \quad u = r, \quad v = t, \quad w = 0,$$

$$u' = 0, \quad v' = 0, \quad w' = s:$$

substituting these values in the equation, we reduce it to

$$rt - s^2 = 0.$$

This is the condition which subsists for every point of developable surfaces, as might have been anticipated, since in the case of this class of surfaces the tangent plane at every point touches them along a straight line. Instances of singular lines of the species which we have been considering occur on the wave-surface, being circles the planes of which are at right angles to the wave-axes. See the *Cambridge Mathematical Journal*, vol. I. p. 83.

CHAPTER XIV.

ON THE CURVATURE OF CURVES IN SPACE.

(260) Let any number of points P, P', P'', P''', ... (fig. 29) be taken in a curve AB in space. Join PP', $P'P''$, $P''P'''$,... by straight lines: these lines will be chords of the curve, and when the number of the points is increased without limit, will ultimately assume ratios of equality with the corresponding elements of the arc of the curve. Produce PP', $P'P''$, indefinitely to points T, T'; PT, $P'T'$, will ultimately be tangents to consecutive points of the curve: and it is evident that the amount of the curvature of the curve in the vicinity of the point P may be properly measured by the ratio of the angle $TP'T'$ to the chord or elemental arc PP', that is, by the rate at which the tangent changes its direction in passing from any point of a curve to a consecutive one.

(261) In the osculating plane $PP'P''$ draw two normals KO, $K'O$, from the middle points K, K', of the chords PP', $P'P''$: these normals will include an angle KOK' equal to the angle $TP'T'$: call this angle $d\epsilon$; the point O, in which the two normals intersect, will be the centre of a circle passing through the three points P, P', P''. This circle is called the *osculating circle* to the curve at the point P, in consequence of having two consecutive elements PP', $P'P''$, in common with the curve. Each of the distances OP, OP', OP'', is a radius of this circle. Supposing the elements PP', $P'P''$, to have been taken equal, then the bent line KPK' will ultimately be an elemental arc ds, both of the curve and of the osculating circle: and, since it subtends at O an angle $d\epsilon$, we shall have in the limit, ρ being the ultimate value of OP,

$$d\epsilon = \frac{ds}{\rho}, \quad \text{or} \quad \frac{d\epsilon}{ds} = \frac{1}{\rho} \;\ldots\ldots\ldots\ldots (1).$$

This result shews that the *curvature of a curve*, as indicated by the ratio $\frac{d\epsilon}{ds}$, varies from point to point of the curve inversely as the radius of the osculating circle: for this reason, this line is called the *radius of curvature* of the curve at the point. The angle $d\epsilon$ is called the *angle of contingence.* The radius of the osculating circle is sometimes called the *radius of absolute curvature* to distinguish it from the *radius of spherical curvature*, which is the radius of a sphere passing through four consecutive points of the curve.

(262) If we suppose the osculating plane $PP'P''$ to revolve through a certain angle about the chord $P'P''$, we shall bring it into the same plane with the consecutive osculating plane $P'P''P'''$, the three elements PP', $P'P''$, $P''P'''$, being thus brought into a single plane. If the plane $PP'P''P'''$ be then turned through a certain angle about $P''P'''$, we shall have four elements of the curve in a single plane; and so on indefinitely. Thus, a curve in space may be in this way reduced to a plane curve. Conversely, by opposite movements, we may change a plane curve into a curve not contained in any one plane The ratio of the angle between two consecutive osculating planes, which we will call $d\theta$, to the length of the elemental arc ds, may be taken as the measure of the *torsion* of a curve in space at any point. This torsion of curves in space may be regarded as a species of curvature, and, as it is of an entirely different nature from the curvature which we have considered above, curves of this class have been called curves of double curvature. The ratio $\frac{d\theta}{ds}$ is never infinite in continuous curves, and is always zero in curves lying in one plane.

(263) To calculate the angle of contingence at any point of a curve in space.

Let x, y, z, be the co-ordinates of any point P (fig. 29) of the curve, ds being the elemental arc PP': let

$$\frac{dx}{ds} = u, \quad \frac{dy}{ds} = v, \quad \frac{dz}{ds} = w,$$

and let u', v', w', denote analogous quantities at the consecutive

point P'. Then, $d\varepsilon$ representing the angle between the tangents at P and P', we have, putting for brevity $d\varepsilon = \varepsilon_1$,

$$\cos \varepsilon_1 = uu' + vv' + ww'.$$

But u', v', w', are the values assumed by u, v, w, when x, y, z, become $x + dx$, $y + dy$, $z + dz$: hence, by Taylor's theorem,

$$u' = u + du + \tfrac{1}{2} d^2u + \frac{1}{2.3} d^3u + \ldots.$$

$$v' = v + dv + \tfrac{1}{2} d^2v + \frac{1}{2.3} d^3v + \ldots.$$

$$w' = w + dw + \tfrac{1}{2} d^2w + \frac{1}{2.3} d^3w + \ldots.$$

and therefore, attending to the relations

$$u^2 + v^2 + w^2 = 1,$$

$$udu + vdv + wdw = 0,$$

$$ud^2u + vd^2v + wd^2w = -(du^2 + dv^2 + dw^2),$$

the two last of which result from the first by differentiation, we see that

$$\cos \varepsilon_1 = 1 - \tfrac{1}{2}(du^2 + dv^2 + dw^2) + \frac{1}{2.3}(ud^3u + vd^3v + wd^3w) + \ldots.$$

but $$\cos \varepsilon_1 = 1 - \tfrac{1}{2}\varepsilon_1^2 \ldots;$$

hence, by taking infinitesimals of the second order on the two sides of the equation, in comparison with which those of the third and higher orders vanish, we get

$$\varepsilon_1^2 = du^2 + dv^2 + dw^2,$$

or, restoring the values of u, v, w,

$$d\varepsilon = \left\{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2\right\}^{\frac{1}{2}} \ldots\ldots (2).$$

Performing the differentiations indicated in the equation (2), we have, dx, dy, dz, ds, being all supposed to be variable,

$$d\varepsilon = \frac{1}{ds^2}\{(dsd^2x - dxd^2s)^2 + (dsd^2y - dyd^2s)^2 + (dsd^2z - dzd^2s)^2\}^{\frac{1}{2}}.$$

Now, squaring the binomials in this expression, we shall obtain, for the coefficient of ds^2,

$$(d^2x)^2 + (d^2y)^2 + (d^2z)^2;$$

for the coefficient of $(d^2s)^2$,

$$dx^2 + dy^2 + dz^2 = ds^2;$$

and, for the coefficient of $ds\, d^2s$,

$$-2\,(dx\,d^2x + dy\,d^2y + dz\,d^2z) = -2ds\,d^2s:$$

hence we see that

$$d\epsilon = \frac{1}{ds}\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}} \ldots\ldots (3).$$

From (3) we get

$$d\epsilon^2\,ds^4 = \{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}\,ds^2 - ds^2\,(d^2s)^2;$$

and therefore, since

$$ds^2 = dx^2 + dy^2 + dz^2,$$

and $$ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z,$$

we have

$$\begin{aligned} d\epsilon^2 ds^4 &= \{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}\,(dx^2 + dy^2 + dz^2) \\ &\qquad - (dx\,d^2x + dy\,d^2y + dz\,d^2z)^2 \\ &= (dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2, \end{aligned}$$

$$d\epsilon = \frac{1}{ds^2}\,\{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2\}^{\frac{1}{2}} \ldots\ldots (4).$$

It may be remarked, that the three binomials under the radical are the same as the coefficients of $x' - x$, $y' - y$, $z' - z$, in the equation to the osculating plane, Art. (180).

From the different expressions (2), (3), (4), which have been obtained for $d\epsilon$, we may get a variety of formulæ for the radius of curvature at any point of the curve, by virtue of equation (1). Thus, taking the expression for $d\epsilon$ given in (3), we have

$$\rho = \frac{ds^2}{\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}}}.$$

(264) This method of obtaining the radius of curvature serves for the determination of its magnitude, but gives us no information respecting its position. We shall proceed, therefore, to develop another method of investigation, which will determine at once both the length of the radius of curvature and the co-ordinates of the centre of the osculating circle.

The normals KO, $K'O$ (fig. 29) are the intersections of the osculating plane $PP'P''$ with two consecutive normal planes:

hence the determination of the centre of curvature O is coincident with the determination of the point of intersection of these three planes.

The equation to the osculating plane at the point P, the co-ordinates of which are x, y, z, will be (Art. 180)

$$A(x'-x)+B(y'-y)+C(z'-z)=0 \;\ldots\; (5);$$

where

$$A = dy\,d^2z - dz\,d^2y,$$
$$B = dz\,d^2x - dx\,d^2z,$$
$$C = dx\,d^2y - dy\,d^2x.$$

The equation to the normal plane at the same point is (Art. 178)

$$(x'-x)\,dx+(y'-y)\,dy+(z'-z)\,dz=0 \;\ldots\; (6).$$

The equation to the normal plane at the consecutive point $x+dx$, $y+dy$, $z+dz$, will be

$$(x'-x)(dx+d^2x)+(y'-y)(dy+d^2y)+(z'-z)(dz+d^2z) - (dx^2+dy^2+dz^2)=0;$$

and therefore, at the point in which the two consecutive normal planes intersect,

$$(x'-x)\,d^2x+(y'-y)\,d^2y+(z'-z)\,d^2z-ds^2=0 \;\ldots\; (7).$$

Multiplying (5), (6), (7), by $1, \lambda, \lambda'$, respectively, λ and λ' being indeterminate multipliers, adding together the resulting equations, and equating to zero the coefficients of $y'-y$ and $z'-z$ in the final equation, we have

$$(x'-x)(A+\lambda\,dx+\lambda'\,d^2x)=\lambda'\,ds^2 \;\ldots\; (8),$$
$$B+\lambda\,dy+\lambda'\,d^2y=0 \;\ldots\; (9),$$
$$C+\lambda\,dz+\lambda'\,d^2z=0 \;\ldots\; (10).$$

From (9) and (10) we have

$$Bdz-Cdy=\lambda'(dy\,d^2z-dz\,d^2y)=\lambda' A,$$

and

$$Cd^2y-Bd^2z=\lambda(dy\,d^2z-dz\,d^2y)=\lambda A;$$

and therefore, from (8),

$$(x'-x)\{A^2+dx(Cd^2y-Bd^2z)+d^2x(Bdz-Cdy)\}=ds^2(Bdz-Cdy),$$

or

$$(x'-x)(A^2+B^2+C^2)=ds^2(Bdz-Cdy).$$

Similarly we have

$$(y'-y)(A^2+B^2+C^2)=ds^2(Cdx-Adz),$$
$$(z'-z)(A^2+B^2+C^2)=ds^2(Ady-Bdx).$$

Hence, observing that

$$\rho^2=(x'-x)^2+(y'-y)^2+(z'-z)^2,$$

we have

$$\rho(A^2+B^2+C^2)=ds^2\{(Bdz-Cdy)^2+(Cdx-Adz)^2+(Ady-Bdx)^2\}^{\frac{1}{2}}.$$

Developing the squares of the three binomials under the radical, the terms multiplied by A^2, B^2, C^2, will evidently be

$$A^2(dy^2+dz^2)=A^2(ds^2-dx^2),$$
$$B^2(dz^2+dx^2)=B^2(ds^2-dy^2),$$
$$C^2(dx^2+dy^2)=C^2(ds^2-dz^2):$$

thus the radical becomes

$$\{(A^2+B^2+C^2)\,ds^2-(Adx+Bdy+Cdz)^2\}^{\frac{1}{2}}.$$

But, restoring to A, B, C, their values, we see that

$$Adx+Bdy+Cdz$$

is identically zero: hence the radical becomes

$$(A^2+B^2+C^2)^{\frac{1}{2}}\,ds;$$

and for the radius of curvature we get

$$\rho=\frac{ds^3}{(A^2+B^2+C^2)^{\frac{1}{2}}}$$
$$=\frac{ds^3}{\{(dy\,d^2z-dz\,d^2y)^2+(dz\,d^2x-dx\,d^2z)^2+(dx\,d^2y-dy\,d^2x)^2\}^{\frac{1}{2}}},$$

a result which agrees with the formulæ (1) and (4).

If in the numerators of the expressions for the co-ordinates x', y', z', of the centre of curvature, we restore the values of A, B, C, we shall have, in the case of the first,

$$\begin{aligned}Bdz-Cdy&=d^2x\,(dy^2+dz^2)-dx\,(dy\,d^2y+dz\,d^2z)\\&=d^2x\,(ds^2-dx^2)-dx\,(ds\,d^2s-dx\,d^2x)\\&=d^2x\,ds^2-dx\,ds\,d^2s,\end{aligned}$$

analogous expressions resulting for the second and third: we shall have, therefore,

$$x' - x = ds^3 \cdot \frac{ds\, d^2x - dx\, d^2s}{A^2 + B^2 + C^2} = \rho^2 \frac{d\left(\dfrac{dx}{ds}\right)}{ds} \dots\dots (11),$$

$$y' - y = ds^3 \cdot \frac{ds\, d^2y - dy\, d^2s}{A^2 + B^2 + C^2} = \rho^2 \frac{d\left(\dfrac{dy}{ds}\right)}{ds} \dots\dots (12),$$

$$z' - z = ds^3 \cdot \frac{ds\, d^2z - dz\, d^2s}{A^2 + B^2 + C^2} = \rho^2 \frac{d\left(\dfrac{dz}{ds}\right)}{ds} \dots\dots (13).$$

Let α, β, γ, denote the angles which the radius of curvature ρ, estimated in the direction from x, y, z, to x', y', z', forms with the co-ordinate axes: then

$$\cos \alpha = \frac{x' - x}{\rho} = \rho \frac{d\left(\dfrac{dx}{ds}\right)}{ds},$$

$$\cos \beta = \frac{y' - y}{\rho} = \rho \frac{d\left(\dfrac{dy}{ds}\right)}{ds},$$

$$\cos \gamma = \frac{z' - z}{\rho} = \rho \frac{d\left(\dfrac{dz}{ds}\right)}{ds}.$$

From these three formulæ, observing that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

we obtain

$$\frac{ds}{\rho} = \left\{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2\right\}^{\frac{1}{2}} \dots\dots\dots (14),$$

a result which is in agreement with the formulæ (1) and (2).

(265) The determination of the centre and radius of the osculating circle may be effected also by the following simple method.*

Let PQ, QR, (fig. 30) be two consecutive elements of the curve, and let us suppose them to be equal, which is the same thing as considering ds constant or taking s for the independent variable. Complete the parallelograms $PQRS$, $SQTR$, and let M, N, M', N', be the projections of P, Q, R, T, upon the

* This method was given by Mr. Archibald Smith, of Trinity College, in the *Cambridge Mathematical Journal* for February 1838.

axis of x. The centre O of the osculating circle, which passes through the three points P, Q, R, will evidently lie in the line QS produced. Let Q' be the extremity of the diameter through Q: join RQ' and PR: QS will be bisected by PR in a point V. Then, QRV, $QQ'R$, being right-angled triangles, we have

$$QR^2 = QQ' \times QV,$$

or, putting $QO = \rho, \quad QS = \lambda, \quad QR = ds,$

$$ds^2 = \rho\lambda.$$

Let α, β, γ, be the angles which QO makes with the co-ordinate axes; then

$$QS \cos \alpha = RT \cos \alpha = M'N':$$

but $M'N' = NN' - NM' = MN - NM' = d^2x:$

hence $\lambda \cos \alpha = d^2x,$

and therefore $$\cos \alpha = \rho \frac{d^2x}{ds^2}.$$

Similarly $$\cos \beta = \rho \frac{d^2y}{ds^2},$$

$$\cos \gamma = \rho \frac{d^2z}{ds^2}.$$

Hence, observing that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

we obtain $$\frac{1}{\rho} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}.$$

If we change the independent variable from s to any other quantity, we shall arrive at the formulæ obtained above for ρ. Thus, the formula (14) results immediately from this expression for ρ, by introducing the alteration corresponding to the supposition that ds is no longer constant.

Again, x', y', z', being the co-ordinates of the centre of curvature, we get, putting for ρ its value just obtained in the expressions for $\cos \alpha$, $\cos \beta$, $\cos \gamma$,

$$x' - x = \rho \cos \alpha = \rho^2 \frac{d^2x}{ds^2},$$

$$y' - y = \rho \cos \beta = \rho^2 \frac{d^2y}{ds^2},$$

$$z' - z = \rho \cos \gamma = \rho^2 \frac{d^2z}{ds^2}.$$

(266) To calculate an expression for $d\theta$, the angle of torsion.

The equation to the osculating plane at the point x, y, z, being

$$Ax' + By' + Cz' = D,$$

the equation to the consecutive osculating plane will be

$$A'x' + B'y' + C'z' = D',$$

where $\quad A' = A + dA, \quad B' = B + dB, \quad C' = C + dC.$

Now, $d\theta$ being the angle between these two planes, which for brevity we will call θ_1,

$$\cos\theta_1 = \frac{AA' + BB' + CC'}{(A^2 + B^2 + C^2)^{\frac{1}{2}} \cdot (A'^2 + B'^2 + C'^2)^{\frac{1}{2}}};$$

and therefore

$$\sin^2\theta_1 = \frac{(BC' - CB')^2 + (CA' - AC')^2 + (AB' - BA')^2}{(A^2 + B^2 + C^2) \cdot (A'^2 + B'^2 + C'^2)}$$

$$= \frac{(BdC - CdB)^2 + (CdA - AdC)^2 + (AdB - BdA)^2}{(A^2 + B^2 + C^2)^2}.$$

But $\quad B = dz\,d^2x - dx\,d^2z, \quad dB = dz\,d^3x - dx\,d^3z,$

$\quad C = dx\,d^2y - dy\,d^2x, \quad dC = dx\,d^3y - dy\,d^3x,$

and therefore

$$BdC - CdB = dx\,\{dx\,(d^2y\,d^3z - d^2z\,d^3y) + dy\,(d^2z\,d^3x - d^2x\,d^3z) + dz\,(d^2x\,d^3y - d^2y\,d^3x)\},$$

analogous expressions existing for the two other binomial terms in the numerator of the formula for $\sin^2\theta$.

Hence, putting $d\theta$ for $\sin\theta_1$, and ds^2 for $dx^2 + dy^2 + dz^2$, and taking the square root, we have

$$\frac{d\theta}{ds} = \frac{dx\,(d^2y\,d^3z - d^2z\,d^3y) + dy\,(d^2z\,d^3x - d^2x\,d^3z) + dz\,(d^2x\,d^3y - d^2y\,d^3x)}{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2} \quad \ldots\ldots (15),$$

which is a measure of the rate of torsion at any point of the curve.

(267) *Points of Inflected Torsion.* The total amount of the torsion of a curve, as we pass along any proposed length of the arc, is equivalent to an angle θ, that is, to the sum of all the successive values of $d\theta$: if the torsion, after having for a certain space taken place in one direction, then assume an opposite course, the torsion at the point of the curve where this change

takes place may be said to be *inflected*, and the point itself may be termed a *point of inflected torsion*. In passing along the arc through such a point, the measure of torsion must evidently change sign. Hence, by formula (15), we see that the condition for a *point of inflected torsion* coincides with a change of sign in the expression

$$dx\,(d^2y\,d^3z - d^2z\,d^3y) + dy\,(d^2z\,d^3x - d^2x\,d^3z)$$
$$+ dz\,(d^2x\,d^3y - d^2y\,d^3x) \dots . \quad (16).$$

If the change of sign take place so that the expression (15) pass through a zero value, the change in the character of the torsion will be continuous: if this expression pass through infinity, the change of torsion will be abrupt, two consecutive osculating planes including an angle of $-180°$. The point will therefore be cuspidal.

(268) *Points of Suspended and of Infinite Torsion.* The expression for the measure of the rate of torsion given in (15), may be zero at a certain point, although it may not change sign as we pass through the point. The torsion does not in this case change its character, but is merely stationary for a small portion of the arc: the point may therefore be called a point of *suspended torsion.* If the expression for the measure of the rate of torsion be infinite, and there be no change of sign, the increase of the torsion will be abrupt, and the angle between two consecutive osculating planes will be $+180°$: such a point may be called, in relation to the rate of torsion, a *point of infinite torsion.*

If the expression (16) be satisfied identically by the equations to the curve, for all simultaneous co-ordinates, the curve will lie entirely in one plane; a conclusion which agrees with Art. (145).

(269) *Points of Inflected Curvature.* If, as we pass along a particular portion of a curve, the angle $d\epsilon$ lie first on one side of the tangent at each point, and afterwards on the other, it is evident that the nature of the curvature undergoes alteration, concavity and convexity being interchanged. This will be clear on inspecting fig. (31), where three consecutive elements, PP', $P'P''$, $P''P'''$, are drawn, the angle $d\epsilon$, in the osculating

plane $PP'P''$, being below the tangent at P, and, in the osculating plane $P'P''P'''$, above the tangent at P'. The point where the change takes place may be called a *point of inflected curvature.*

Suppose first, that the change of the character of the curvature is continuous: then, since $\frac{d\varepsilon}{ds}$ must change sign through zero, we see, from formula (3), supposing s to be the independent variable, that

$$\frac{d^2x}{ds^2} = 0, \quad \frac{d^2y}{ds^2} = 0, \quad \frac{d^2z}{ds^2} = 0 \ldots\ldots (17).$$

If we suppose x to be the independent variable, then, from formula (4), we see that, as necessary conditions,

$$\frac{\dfrac{d^2y}{dx^2}}{\dfrac{ds^3}{dx^3}} = 0, \quad \frac{\dfrac{d^2z}{dx^2}}{\dfrac{ds^3}{dx^3}} = 0 \ldots\ldots\ldots\ldots (18).$$

If $\frac{ds}{dx}$ be finite, these two conditions evidently reduce themselves to $\quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^2z}{dx^2} = 0.$

When $\frac{ds}{dx}$ is infinite, the conditions will occasionally be satisfied when $\quad \frac{d^2y}{dx^2} = \infty, \quad \frac{d^2z}{dx^2} = \infty.$

If the change of the nature of the curvature be abrupt, then $\frac{d\varepsilon}{ds}$ will pass through ∞, and two consecutive elements of the curve will be inclined to each other at an angle of -180°. Hence, in one or more of the equations (17), or of the equivalent equations (18), 0 must be replaced by ∞.

(270) *Points of Suspended and of Infinite Curvature.* If a change of sign do not take place in the value of $\frac{d\varepsilon}{ds}$ when it passes through zero, the curvature is merely suspended; and if, without changing sign, its value pass through infinity,

the rate of curvature is infinite. Such points may be called respectively *points of suspended* and *points of infinite curvature.*

(271) *Points of Inflected* and of *Suspended Torsion* are ordinarily comprehended under the appellation of *points of simple inflection;* an essential property of such points being the coincidence of two consecutive osculating planes.

Points of Inflected and of *Suspended Curvature* are ordinarily denoted by the common name of *points of double inflection;* two consecutive elements at such points lie in a single line. These points have been called points of double inflection, because their existence involves that of points of simple inflection. I have adopted different appellations for such singular points, because I think the ordinary terms do not correspond with sufficient distinctness to the true geometrical peculiarities of the points.

(272) There is an important distinction between plane curves and curves of double curvature, in regard to their radii of curvature. In plane curves the radii of curvature intersect each other consecutively, the locus of these intersections being the evolute, to which all the radii are tangents; while in curves of double curvature the radii of the osculating circles do not meet each other consecutively. Through the middle points $K, K', K'',\ldots$ of the several elements $PP', P'P'', P''P''',\ldots.$ of the curve $PP'P'',\ldots.$(fig. 32), draw normal planes $L, L', L'',\ldots.$ intersectting each other consecutively in the straight lines $AB, A'B',\ldots$ and thus forming a developable surface, the envelop of all the planes. If we cut the planes $L, L',\ldots.$ by the osculating plane $PP'P''$, which is at right angles to both of them, the lines of intersection will be the normals KC and $K'C$, perpendicular to AB, and of which the former will be the radius of curvature of the curve at the point P. In the same way, cutting the normal planes L' and L'' by the osculating planes $P'P''P'''$, we shall have, for the sections, the two normals $K'C'$ and $K''C'$, perpendicular to $A'B'$, the former of which will be the radius of curvature at the point P'. Now it is evident that the radius $K'C'$ does not coincide with the other normal $K'C$, because these two lines are the intersections of the same plane L' with two

different osculating planes: hence $K'C'$ will meet AB in a point I different from C, and consequently the two radii of curvature, KC' and $K'C'$, situated in the planes L and L', have not a common point at the intersection of these two planes: it follows, therefore, that these two radii do not meet.

From what we have said, then, it appears that the centres of curvature C, C', C'',.... do not result from the successive intersections of the radii KC, $K'C'$, $K''C''$,.... and that consequently these radii are not tangents to the locus of these points: the radii of curvature cannot therefore be regarded as formed by the unwrapping of a thread wound about the curve $CC'C''$....: in other words, the $CC'C''$,.... is not an evolute of the curve $PP'P''$....whenever this latter curve has double curvature.

(273) Although the locus of the centres of the osculating circles is not an evolute of $PP'P''$.... ; yet, as Monge has shewn, this curve may be shewn to possess an infinite number of evolutes. In fact, if in L, the first of the normal planes, we draw arbitrarily a straight line KD (fig. 32), which will always be normal to the proposed curve; and then, through the D and K', draw another straight line $K'DD'$, which will lie in L', the second normal plane; then a third line $K''D'D''$ situated in the plane P'', and so on successively, we shall obtain, by the successive intersections of these normals, a curve $DD'D''$.... to which these normals will be tangents. The curve $PP'P''$....may evidently be described by unwrapping a string wound about the curve $DD'D''$,....which will thus be an evolute of the former. In proof of this, it is sufficient to observe that the portions DK and DK' of the tangents to $DD'D''$....are equal to each other, or that the point D is at the same distance from the three points M, M', M''; for it is clear that each point of the line AB, which is the intersection of the two planes L and L' drawn at right angles to the elements PP' and $P'P''$ through their middle points, must be at the same distance from P, from P', and from P''. Moreover, since the first normal KD was drawn arbitrarily in the plane L, we may, by varying the direction of this normal, obtain an infinite variety of evolutes all situated on the enveloping surface of the normal planes.

CHAPTER XV.

ON THE CURVATURE OF SURFACES.

(274) A plane is said to be *normal* to a surface when it contains a normal line. If at any proposed point of a surface, a series of normal planes be drawn, the radii of curvature of the various normal sections of the surface at the point will generally vary: from a comparison of the curvatures of the different normal sections, we shall arrive at a conception of the nature of the curvature of the surface around the point in question.

The radius of curvature at any point of a plane normal section of a surface is determined by the intersection of the normal to the surface at the point with the normal plane at a consecutive point of the curve of section.

(275) Let the equation to the surface be

$$F(x, y, z) = 0:$$

then, adopting the notation of Chapter XIII., we shall have, for the equations to the normal at the point (x, y, z),

$$\frac{x' - x}{U} = \frac{y' - y}{V} = \frac{z' - z}{W} \quad \ldots\ldots\ldots\ldots (1).$$

Now the direction-cosines of any tangent line are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds},$$

and the tangent line may be taken as that which touches the curve of section at the point where the normal is drawn. Hence the equation to the normal plane, at the point (x, y, z), will be

$$(x' - x)\, dx + (y' - y)\, dy + (z' - z)\, dz = 0 \ldots\ldots (2).$$

Since, by differentiating the equation to the surface, we get

$$Udx + Vdy + Wdz = 0 \ldots\ldots\ldots\ldots (3),$$

it is evident that the line (1) lies in the plane (2): the intersection of (1) with the consecutive normal plane will therefore lie in the line of intersection of (2) with the consecutive normal plane. The equations to this line of intersection are (2), and an equation obtained by differentiating (2) with respect to x, y, z, viz.

$$(x'-x)\,d^2x + (y'-y)\,d^2y + (z'-z)\,d^2z = dx^2 + dy^2 + dz^2 = ds^2 \ldots (4).$$

The equations (1) and (4), taken together, determine x', y', z', the co-ordinates of the centre of curvature of the section, and the radius of curvature is the distance between that point and (x, y, z). From (1) and (4), there is

$$\frac{x'-x}{U} = \frac{y'-y}{V} = \frac{z'-z}{W} = \frac{ds^2}{Ud^2x + Vd^2y + Wd^2z} \ldots (5):$$

hence also, ρ denoting the radius of curvature,

$$\rho = \pm\left\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\right\}^{\frac{1}{2}} = \pm\frac{(U^2 + V^2 + W^2)^{\frac{1}{2}}\,ds^2}{Ud^2x + Vd^2y + Wd^2z} \ldots (6),$$

which is one expression for the radius of curvature.

(276) We may eliminate the second differentials d^2x, d^2y, d^2z, from the equations (5) and (6) in the following manner. Differentiating (3), we have

$$Ud^2x + Vd^2y + Wd^2z + udx^2 + vdy^2 + wdz^2$$
$$+ 2u'dydz + 2v'dzdx + 2w'dxdy = 0:$$

availing ourselves of this equation, and putting

$$\frac{dx}{ds} = l, \quad \frac{dy}{ds} = m, \quad \frac{dz}{ds} = n,$$

we readily transform the equations (5) and (6) into

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw'$$
$$= \frac{U}{x-x'} = \frac{V}{y-y'} = \frac{W}{z-z'} \ldots\ldots\ldots\ldots (7),$$

and $$\rho = \frac{\mp(U^2 + V^2 + W^2)^{\frac{1}{2}}}{l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw'} \ldots\ldots (8).$$

(277) There is no condition which will enable us to select one of the two signs introduced by the radical into the expression

for ρ in preference to the other, although at each point of the surface for an assigned normal plane ρ must have some determinate position. The ambiguity of sign indicates that it is quite arbitrary which side of the tangent plane we adopt for the positive direction of ρ. The actual position of ρ can be obtained only from the equations (7), the equation (8) serving merely to determine its magnitude.

(278) The equations (7) and (8) may easily be transformed so as to involve the partial differential coefficients of z with respect to x and y, instead of the partial differential coefficients of $F(x, y, z)$ with respect to x, y, z.

Conceive the equation to the surface to be reduced to the form

$$F(x, y, z) = f(x, y) - z = 0,$$

z being thus rendered explicit. Then it is easily seen that

$$U = \frac{dz}{dx} = p, \quad V = \frac{dz}{dy} = q, \quad W = -1,$$

$$u = \frac{d^2z}{dx^2} = r, \quad v = \frac{d^2z}{dy^2} = t, \quad w = 0,$$

$$u' = 0, \quad v' = 0, \quad w' = \frac{d^2z}{dxdy} = s.$$

The equations (7) and (8) will therefore become

$$l^2r + 2lms + m^2t$$

$$= \frac{p}{x - x'} = \frac{q}{y - y'} = \frac{-1}{z - z'}$$

and $$\rho = \frac{\mp(1 + p^2 + q^2)^{\frac{1}{2}}}{l^2r + 2lms + m^2t}.$$

(279) For all normal sections passing through x, y, z, the quantities $$U, V, W, u, v, w, u', v', w',$$

are constant; but the expressions for $x - x'$, $y - y'$, $z - z'$, ρ, will change as l, m, n, vary; the variation of l, m, n, taking place in accordance with the two conditions

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots (9),$$

$$lU + mV + nW = 0 \ldots\ldots\ldots\ldots (10);$$

the latter of which expresses the perpendicularity of the tangent line to the normal.

Since, at any assigned point of the surface, the quantities $x - x'$, $y - y'$, $z - z'$, in consequence of the variations of l, m, n, have changes of magnitude, it is possible that they may likewise experience changes of sign. Should a change of sign take place in the value of any one of them, it will take place simultaneously in the values of all, the expression

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw',$$

which is common to all of them, being the only variable element in their values. Such changes of sign indicate that the centre of the circle of curvature must lie for different sections in opposite directions from the point (x, y, z), or that the surface in the vicinity of the point lies partly on one side and partly on the other side of the tangent plane. We proceed to ascertain under what conditions these changes of sign can take place.

Put $$\frac{U}{x - x'} = \frac{V}{y - y'} = \frac{W}{z - z'} = \frac{A}{R^2},$$

A being a constant: hence we see that a change of sign in $x - x'$, $y - y'$, $z - z'$, will take place simultaneously with a change of sign in R^2: but, from (7), we see that

$$R^2 (l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw') = A;$$

or, putting $Rl = x_1$, $Rm = y_1$, $Rn = z_1$,

$$ux_1^2 + vy_1^2 + wz_1^2 + 2u'y_1z_1 + 2v'z_1x_1 + 2w'x_1y_1 = A \dots\dots (11):$$

we have also, from (10),

$$Ux_1 + Vy_1 + Wz_1 = 0 \dots\dots\dots\dots (12).$$

Thus we see that a change of sign in the values of $x - x'$, $y - y'$, $z - z'$, is coincident with a change of sign in the square of the radius-vector of a central conic section, of which the equations are (11) and (12). If the conic section be an ellipse, all its radii are possible, and therefore their squares are always positive, and the values of $x - x'$, $y - y'$, $z - z'$, have always the same sign. If the conic section be an hyperbola, some of the radii are possible and some impossible; their squares may, therefore, be

positive or negative, and the changes of sign which we are considering may take place.

The nature of the conic section will be best seen by supposing the surface to be referred to the tangent plane at the point (x, y, z) as the plane of xy, and then the equation to the conic section is reduced to

$$ux_1^2 + 2w'x_1y_1 + vy_1^2 = A,$$
$$z_1 = 0.$$

The equation will be an ellipse if

$$uv - w'^2 > 0,$$

and an hyperbola, if $\quad uv - w'^2 < 0.$

If it should happen that $uv - w'^2 = 0,$

then the hyperbola will degenerate into two straight lines: also

$$R^2 = \frac{Au}{(lu + mw')^2} = \frac{Av}{(lw' + mv)^2},$$

so that R^2 never changes sign: if the relation between l and m be such that $\quad lu + mw' = 0, \quad$ or $\quad lw' + mv = 0,$

these two relations being really one and the same, then $x - x'$, $y - y'$, $z - z'$, ρ, all become infinite.

The relation $\quad uv - w'^2 = 0$

is satisfied in the instance of developable surfaces, the infinite values of $x - x'$, &c. having relation to sections along the generating lines of the surface.

(280) To find the greatest and least radii of curvature of the normal sections at any point of a surface.

If we put $\quad \mp (U^2 + V^2 + W^2)^{\frac{1}{2}} = P,$

we have to make

$$\frac{P}{\rho} = l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' \ldots . (13)$$

a maximum or minimum; $l, m, n,$ being supposed to vary in agreement with the relations (9) and (10). From (10), we have

$$m^2V^2 + 2mnVW + n^2W^2 = l^2U^2,$$

$$2mn = \frac{l^2U^2 - m^2V^2 - n^2W^2}{VW}:$$

similarly
$$2nl = \frac{m^2V^2 - n^2W^2 - l^2U^2}{WU},$$
$$2lm = \frac{n^2W^2 - l^2U^2 - m^2V^2}{UV}.$$

Substituting these expressions for $2mn$, $2nl$, $2lm$, in (13), we get
$$\frac{P}{\rho} = Hl^2 + Km^2 + Ln^2 \text{ .... } (14),$$
where
$$\left.\begin{aligned} H &= u + \frac{U}{VW}(Uu' - Vv' - Ww') \\ K &= v + \frac{V}{WU}(Vv' - Ww' - Uu') \\ L &= w + \frac{W}{UV}(Ww' - Uu' - Vv') \end{aligned}\right\} \text{ ...... } (15).$$

That ρ may be a maximum or minimum, we must equate the differential of $\frac{P}{\rho}$ to zero: hence, from (9), (10), (14), we have
$$ldl + mdm + ndn = 0,$$
$$Udl + Vdm + Wdn = 0,$$
$$Hldl + Kmdm + Lndn = 0.$$

Eliminating dl, dm, dn, by indeterminate multipliers, we have
$$\left.\begin{aligned} (H + \lambda)\, l - \mu U &= 0 \\ (K + \lambda)\, m - \mu V &= 0 \\ (L + \lambda)\, n - \mu W &= 0 \end{aligned}\right\} \text{ .... } (16).$$

Multiplying these three equations in order by l, m, n, and adding, we have, by (9), (10), (14),
$$\frac{P}{\rho} + \lambda = 0.$$

Hence, from the equations (16),
$$\left.\begin{aligned} l &= \frac{\mu U}{H - \frac{P}{\rho}} \\ m &= \frac{\mu V}{K - \frac{P}{\rho}} \\ n &= \frac{\mu W}{L - \frac{P}{\rho}} \end{aligned}\right\} \text{ .... } (17).$$

Multiplying these equations in order by U, V, W, and adding, we have, by (10),

$$\frac{U^2}{H-\frac{P}{\rho}}+\frac{V^2}{K-\frac{P}{\rho}}+\frac{W^2}{L-\frac{P}{\rho}}=0 \ldots\ldots (18).$$

This quadratic equation determines, in terms of the quantities U, V, W, H, K, L, P, which are known for any point of the surface, the greatest and least values of ρ at that point. By substituting either of these values in (17), we determine the ratios $l : m : n$, which give the position of the corresponding normal section.

(281) These equations also enable us to prove that the normal sections of greatest and least curvature are at right angles to one another. Thus, if l_1, m_1, n_1, and l_2, m_2, n_2, be the values of l, m, n, corresponding to ρ_1, ρ_2, the greatest and least values of ρ, the equations (17) and (18) must be satisfied when each of these systems of values is substituted for l, m, n, and ρ. Hence, writing down (18) for each value of ρ, and subtracting one equation from the other, we get

$$\left(\frac{P}{\rho_1}-\frac{P}{\rho_2}\right)\left\{\frac{U^2}{\left(H-\frac{P}{\rho_1}\right)\left(H-\frac{P}{\rho_2}\right)}+\frac{V^2}{\left(K-\frac{P}{\rho_1}\right)\left(K-\frac{P}{\rho_2}\right)}+\frac{W^2}{\left(L-\frac{P}{\rho_1}\right)\left(L-\frac{P}{\rho_2}\right)}\right\}=0.$$

Hence, if ρ_1 and ρ_2 be different, the second factor must be equal to zero, or, which is the same thing, on account of equations (17),

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0;$$

which shews that the sections are at right angles to one another. The normal sections of greatest and least curvature at any point of the surface are called the *principal sections*, and the radii of curvature the *principal radii of curvature*.*

* The investigations which I have given for the determination of the principal radii of curvature and of the positions of the principal sections, were communicated to me by Mr. Thomson, of St. Peter's College.

(282) In the case of an ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

we have

$$U=\frac{2x}{a^2},\quad V=\frac{2y}{b^2},\quad W=\frac{2z}{c^2},$$

$$u=\frac{2}{a^2},\quad v=\frac{2}{b^2},\quad w=\frac{2}{c^2},$$

$$u'=0,\quad v'=0,\quad w'=0,$$

$$P=\pm 2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}=\frac{2}{p},$$

p being the perpendicular from the origin on the tangent plane. Hence the equation (18) gives us

$$\frac{x^2}{a^2(p\rho-a^2)}+\frac{y^2}{b^2(p\rho-b^2)}+\frac{z^2}{c^2(p\rho-c^2)}=0.$$

From the last term of this quadratic when cleared of its fractional form, which is equal to $\frac{a^2b^2c^2}{p^4}$, it appears that the product of the greatest and least radii of curvature is constant for all points for which p is constant. The equation for ρ may be put into another remarkable form: for if we write it thus,

$$\frac{x^2}{a^2\left(1-\frac{a^2}{p\rho}\right)}+\frac{y^2}{b^2\left(1-\frac{b^2}{p\rho}\right)}+\frac{z^2}{c^2\left(1-\frac{c^2}{p\rho}\right)}=0,$$

and subtract it from the equation to the ellipsoid, we see that

$$\frac{x^2}{a^2-p\rho}+\frac{y^2}{b^2-p\rho}+\frac{z^2}{c^2-p\rho}=1:$$

this is the equation to a concentric surface of the second order, which will be also confocal, since, if a', b', c', be its semi-axes,

$$a'^2=a^2-p\rho,\quad b'^2=b^2-p\rho,\quad c'^2=c^2-p\rho,$$

and therefore

$$a'^2-b'^2=a^2-b^2,\quad b'^2-c'^2=b^2-c^2,\quad c'^2-a'^2=c^2-a^2.$$

(283) The perpendicularity of the principal sections may be concisely established also by the following reasoning.

Putting $n = 0$ in the equation (13), which amounts to taking the tangent plane at the point in question for the plane of xy, we have

$$\frac{P}{\rho} = l^2u + 2lmw' + m^2v \ldots\ldots (19),$$

l and m being subject to the condition

$$l^2 + m^2 = 1 \ldots\ldots\ldots\ldots (20).$$

Differentiating these two equations with regard to l and m, and putting $d\rho = 0$, we get, by means of an indeterminate multiplier λ,

$$lu + mw' = \lambda l \ldots\ldots\ldots (21),$$

$$mv + lw' = \lambda m \ldots\ldots\ldots (22);$$

whence

$$(u - v)\, lm + w'(m^2 - l^2) = 0,$$

$$\frac{m^2}{l^2} + \frac{u-v}{w'} \cdot \frac{m}{l} - 1 = 0:$$

hence, l_1, m_1, and l_2, m_2, being the values of l, m, for the principal sections, we see that

$$\frac{m_1}{l_1} \cdot \frac{m_2}{l_2} = -1,$$

which is the condition for the perpendicularity of the sections.

(284) To prove that the curvature of any normal section is equal to the sum of the curvatures of the two principal sections, multiplied respectively by the squares of the cosines of the angles which the principal planes make with the normal plane.

Multiplying (21) and (22) by l and m respectively, and adding, we get by (19),

$$\lambda = \frac{P}{\rho};$$

and therefore, eliminating l and m between (21) and (22),

$$\left(u - \frac{P}{\rho}\right)\left(v - \frac{P}{\rho}\right) = w'^2 \ldots\ldots (23).$$

Let ρ_1, ρ_2, denote the principal radii of curvature, and suppose that the planes of xz, yz, coincide with the principal planes. Then, from (19), putting $l = 1$, $m = 0$, $\rho = \rho_1$, simultaneously, we have

$$\frac{P}{\rho_1} = u:$$

similarly, putting $l = 0$, $m = 1$, $\rho = \rho_2$,

$$\frac{P}{\rho_2} = v.$$

Thus u and v are the roots of the equation (23), which is a quadratic in $\frac{P}{\rho}$: hence it follows that $w' = 0$. We obtain, therefore, from (19),

$$\frac{P}{\rho} = l^2 u + m^2 v = P\left(\frac{l^2}{\rho_1} + \frac{m^2}{\rho_2}\right);$$

or, if $l = \cos a$, $m = \sin a$,

$$\frac{1}{\rho} = \frac{\cos^2 a}{\rho_1} + \frac{\sin^2 a}{\rho_2} \ldots\ldots\ldots (24),$$

which establishes our proposition.

(285) We have shewn that if the planes of xz, yz, coincide with the principal sections, we must have, as a necessary condition, $w' = 0$. It may easily be seen that this condition is also sufficient; for, putting $w' = 0$ in (21) and (22), we get

$$lu = \lambda l, \quad mv = \lambda m,$$

and therefore $\quad mlu = mlv;$

an equation which may be satisfied by $l = 0$, $m = 1$, or by $l = 1$, $m = 0$.

(286) The formula (24) may be established also in the following manner.*

Let (l_1, m_1, n_1), (l_2, m_2, n_2), be the direction-cosines of any two lines through a point P, at right angles to one another, in a plane of which the direction-cosines are proportional to U, V, W. Then

$$Ul_1 + Vm_1 + Wn_1 = 0 \ldots\ldots\ldots (a),$$

$$Ul_2 + Vm_2 + Wn_2 = 0 \ldots\ldots\ldots (b).$$

Again, let (l, m, n) be the direction-cosines of a line in the same plane passing through P, and making an angle θ with the line (l_1, m_1, n_1). We have then

$$Ul + Vm + Wn = 0 \ldots\ldots\ldots (c),$$

$$\left.\begin{array}{l} ll_1 + mm_1 + nn_1 = \cos a \\ ll_2 + mm_2 + nn_2 = \sin a \\ l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \end{array}\right\} \ldots\ldots (d).$$

* This method of investigating the formula was communicated to me by Mr. Blackburn, of Trinity College.

Now, λ_1, λ_2, being arbitrary multipliers, $(c) - \lambda_1 (a) - \lambda_2 (b)$ gives

$$l = \lambda_1 l_1 + \lambda_2 l_2,$$
$$m = \lambda_1 m_1 + \lambda_2 m_2,$$

whence

$$n = \lambda_1 n_1 + \lambda_2 n_2;$$

$$ll_1 + mm_2 + nn_2 = \lambda_1 (l_1^2 + m_1^2 + n_1^2) + \lambda_2 (l_1 l_2 + m_1 m_2 + n_1 n_2),$$

or, by (d), $\qquad \cos a = \lambda_1:$

similarly, $\qquad \sin a = \lambda_2.$

Hence

$$\left.\begin{aligned} l &= l_1 \cos a + l_2 \sin a \\ m &= m_1 \cos a + m_2 \sin a \\ n &= n_1 \cos a + n_2 \sin a \end{aligned}\right\} \ldots\ldots\ldots (e),$$

a geometrical theorem.

Let the plane considered be the tangent plane to a surface at a point (x, y, z), and let (l_1, m_1, n_1), (l_2, m_2, n_2), be the direction-cosines of the normals to the principal normal planes. Then, by the formulæ (17),

$$\left(H - \frac{P}{\rho_1}\right) l_1 = \mu U,$$

$$\left(K - \frac{P}{\rho_1}\right) m_1 = \mu V,$$

$$\left(L - \frac{P}{\rho_1}\right) n_1 = \mu W.$$

Hence, by (c) and (d),

$$Hll_1 + Kmm_1 + Lnn_1 = \frac{P}{\rho_1} \cos a,$$

$$Hll_2 + Kmm_2 + Lnn_2 = \frac{P}{\rho_2} \sin a:$$

also, by (14), $\qquad Hl^2 + Km^2 + Ln^2 = \dfrac{P}{\rho}.$

We have therefore, by (e),

$$\frac{\cos^2 a}{\rho_1} + \frac{\sin^2 a}{\rho_2} = \frac{1}{\rho}.$$

(287) The formula (24), the discovery of which is due to Euler, is extremely valuable, as it enables us at once to calculate the radius of curvature of any proposed normal section when the

radii of curvature of the two principal sections are given. If instead of making use of the principal sections of the surface, we were to take any normal planes whatever at any point, it would be necessary to introduce the radii of curvature, R, R', R'', of *three* such sections, and the angles a', a'', contained between them, into the expression for the value of ρ.

Thus, taking one of the normal sections as the plane of xz, if we put in the formula (19), successively,

$$l = 1, \qquad m = 0, \qquad \rho = R;$$
$$l = \cos a', \qquad m = \sin a', \qquad \rho = R';$$
$$l = \cos(a' + a''), \quad m = \sin(a' + a''), \quad \rho = R'';$$

we shall have three equations from which we can find u, w', v, in terms of R, R', R'', a', a''. Hence, by (19), we might determine the value of ρ for any proposed normal section whatever, in terms of the radii of these three normal sections and their contained angles.

(287) If ρ, ρ', be the radii of curvature of any two normal sections at right angles to each other, to prove that

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Putting in the formula (24), $a + \frac{1}{2}\pi$ instead of a, and replacing ρ by ρ', we have
$$\frac{1}{\rho'} = \frac{\sin^2 a}{\rho_1} + \frac{\cos^2 a}{\rho_2}:$$

hence
$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2} \;\ldots\ldots\ldots\ldots\ldots (25);$$

which shews that the sum of the curvatures of any two normal sections at right angles to each other is invariable.

COR. Combining this conclusion with the equation (18), which gives the values of the two quantities ρ_1, ρ_2, we see that, the system of co-ordinates being any whatever,

$$P^3\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) = (K + L)\,U^2 + (L + H)\,V^2 + (H + K)\,W^2.$$

(288) We will proceed to make a few remarks on the equation (24). In the first place, it is important to observe that ρ_1

and ρ_2 are values of ρ in the equation (19) corresponding to the same sign of the radical P: this will be easily seen if, for distinctness of conception, we first take (19) with the positive, and secondly with the negative value of P. We shall find in both cases that in arriving at the relation (24), the same sign, whichever it may be, has been of necessity retained throughout. We must bear in mind, also, that the quantities ρ_1 and ρ_2 are not necessarily symbols of the mere magnitude of the principal radii of curvature, for they may be either both positive, both negative, or the one positive and the other negative. The sign of ρ will depend upon that of ρ_1, ρ_2, and the value of a. As a consequence of (24), it will therefore follow that (25) signifies that the *analytical* sum of the curvatures at any point is constant, the *geometrical* sum not being subject to such limitation, except when ρ_1 and ρ_2 have the same sign.

When ρ_1 and ρ_2 have the same sign, then it is evident that ρ will have always the same sign as either of them: this shews that all the normal sections at the point in question lie, in the neighbourhood of the point, on the same side of the tangent plane; or that the surface is *convex* at the point.

Suppose that ρ_1 is less than ρ_2; then, writing (24) under the forms

$$\frac{1}{\rho} = \frac{1}{\rho_1} - \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \sin^2 a,$$

$$\frac{1}{\rho} = \frac{1}{\rho_2} + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \cos^2 a;$$

it is plain that

$$\frac{1}{\rho_1} > \frac{1}{\rho}, \quad \frac{1}{\rho_2} < \frac{1}{\rho},$$

$$\rho_1 < \rho, \quad \rho_2 > \rho,$$

or that, in absolute value, ρ_1 is a minimum, and ρ_2 a maximum, among all the values of ρ.

If $\rho_1 = \rho_2$; then it is evident, from (24), that

$$\frac{1}{\rho} = \frac{1}{\rho_1} = \frac{1}{\rho_2}, \quad \rho = \rho_1 = \rho_2,$$

whatever be the value of a: thus all the normal sections have at the point the same curvature, and may all equally be regarded

as principal sections. A point of the surface possessed of this peculiarity is called an *umbilicus*. We shall enter, below, more fully into the examination of these points.

Next, let us suppose that the principal radii have opposite signs: for instance, let ρ_1 be positive and ρ_2 negative. In this case the curvatures of the principal sections will be opposite, so that the surface must lie partly above and partly below the tangent plane in the neighbourhood of the point. If we agree to denote by ρ_2 simply its geometrical magnitude, we shall have

$$\frac{1}{\rho} = \frac{\cos^2 \alpha}{\rho_1} - \frac{\sin^2 \alpha}{\rho_2}.$$

Suppose that α' is the least positive value of α which will satisfy the equation
$$\frac{\cos^2 \alpha'}{\rho_1} = \frac{\sin^2 \alpha'}{\rho_2};$$

then, as α increases continuously from $-\alpha'$ up to $2\pi - \alpha'$, it is evident that $\frac{1}{\rho}$ will be zero and therefore ρ infinite for the following values of α, viz.

$$-\alpha', \quad \alpha', \quad \pi - \alpha', \quad \pi + \alpha', \quad 2\pi - \alpha'.$$

It is clear also that ρ will be positive as α varies from $-\alpha'$ to $+\alpha'$, and from $\pi - \alpha'$ to $\pi + \alpha'$; and that it will be negative as α varies from α' to $\pi - \alpha'$, and from $\pi + \alpha'$ to $2\pi - \alpha'$.

If therefore we draw, in the tangent plane at the point, two straight lines inclined at angles $-\alpha'$, $+\alpha'$, or, which comes to the same thing, at angles $\pi - \alpha'$, $\pi + \alpha'$, to the axis of x, these straight lines will be the traces of two normal planes which separate the normal sections, of which the curvature has one direction from those of opposite curvature.

It is easily seen from the formula that ρ_1 is the absolute minimum of all the positive radii, and ρ_2 the absolute minimum of all the negative radii. *Analytically* speaking, ρ_2, affected by its sign minus, is a maximum, being the least of the negative radii. In the case considered above, where ρ_1 and ρ_2 were supposed to have the same sign, one of these quantities is a maximum and the other a minimum, both geometrically and analytically.

In the case of a developable surface, it is easy to see that one of the radii is infinite: thus, from (23),

$$\left(u - \frac{P}{\rho}\right)\left(v - \frac{P}{\rho}\right) = w'^2:$$

but, in a developable surface, as we see by putting U and V both equal to zero in the equation of Art. (255),

$$uv = w'^2:$$

hence

$$\frac{P}{\rho}\left(u + v - \frac{P}{\rho}\right) = 0:$$

thus we see that $\rho_2 = \infty$. Euler's formula is therefore reduced to

$$\frac{1}{\rho} = \frac{\cos^2 \alpha}{\rho_1};$$

a result which shews that ρ has the same sign for all values of α, or that the surface in the vicinity of the point under consideration lies entirely on one side of the tangent plane.

(289) There is a striking analogy between the formula connecting the radius of any normal section with the principal radii, and the relation subsisting between the diameters and axes of an ellipse or an hyperbola. In fact, putting

$$\rho_1 = \frac{a^2}{k}, \quad \rho_2 = \pm\frac{b^2}{k}, \quad \rho = \frac{R^2}{k},$$

the positive or negative sign being taken accordingly as ρ_1, ρ_2, have the same or contrary signs, we have

$$\frac{1}{R^2} = \frac{\cos^2 \alpha}{a^2} \pm \frac{\sin^2 \alpha}{b^2};$$

the equation to a central conic section of which a, b, are the semi-axes, and R a diameter inclined at an angle α to the direction of a. Thus we see that, if the principal axes of the conic section denote the square roots of the principal radii, the diameters will represent the square root of any radius whatever, the corresponding normal section being inclined at the same angle to one of the principal sections as the diameter of the conic section to one of its principal axes.

(290) The curvature of any surface at any of its points may always be assimilated to the curvature of an ellipsoid, or of an hyper-

boloid of one sheet at one of its real vertices. Suppose that in the planes of xz, yz, two ellipses are described, their centres being in the axis of z at a common distance c from the origin, c being a semiaxis of both: let the other semiaxes of the two ellipses be a, b, respectively, a and b being so chosen that $\frac{a^2}{c} = \rho_1$, $\frac{b^2}{c} = \rho_2$. An ellipsoid may be constructed, of which a, b, c, are the semi-axes in magnitude and position. Let R' be the distance of a point in the ellipsoid from its centre, the distance of the point from the plane of xy being c. Then the radius of curvature of the elliptic section of the ellipsoid, made by a normal plane through this point at the origin, will be $\rho' = \frac{R'^2}{c}$. Now the equation to the section of the ellipsoid by a plane through its centre at right angles to the axis of z, gives us

$$\frac{1}{R'^2} = \frac{\cos^2 a}{a^2} + \frac{\sin^2 a}{b^2}:$$

hence, putting for a^2, b^2, their values

$$c\rho_1,\ c\rho_2,$$

respectively, we get

$$\frac{c}{R'^2} = \frac{\cos^2 a}{\rho_1} + \frac{\sin^2 a}{\rho_2}:$$

but $c = \frac{R'^2}{\rho'}$; hence $\quad \frac{1}{\rho'} = \frac{\cos^2 a}{\rho_1} + \frac{\sin^2 a}{\rho_2}.$

But, in relation to the surface,

$$\frac{1}{\rho} = \frac{\cos^2 a}{\rho_1} + \frac{\sin^2 a}{\rho_2}:$$

and therefore $\quad \rho' = \rho.$

Thus we see that the radii of curvature of all normal sections of the surface and of the ellipsoid coincide both in magnitude and in direction; and, accordingly, the ellipsoid has a complete *osculation* with the surface. As we pass from point to point on the surface, the osculating ellipsoid will, of course, generally change, not only in form, but in all the circumstances of position.

If we next suppose ρ_1 to be positive and ρ_2 negative, we must

have, preserving the same notation as in the preceding case, and making the sign of ρ_2 explicit,

$$\frac{a^2}{c} = \rho_1, \quad \frac{b^2}{c} = -\rho_2,$$

c being taken positively as before: thus a will be real and b imaginary; and consequently the ellipse in the plane yz will be changed into an hyperbola, to which the axis of y is a tangent and the origin the vertex, and which lies in the negative direction of the axis of z: the osculating surface will therefore become an hyperboloid of one sheet, of which the ellipse in the plane of xz is the *ellipse de gorge*. The identity of ρ and ρ' for the surface and the hyperboloid may be shewn just as in the case already discussed, of the osculating ellipsoid. The osculating hyperboloid will therefore indicate the exact nature of the curvature of the surface, both in magnitude and direction, for every normal section.

In the case of a developable surface we know that one of the principal radii of curvature, ρ_2 for instance, is infinite. The axis b of the ellipse in the plane of yz will, therefore, become infinite; so that this ellipse will degenerate into two straight lines parallel to the axis of y. The osculating surface will accordingly, in this particular case, degenerate into a right cylinder upon the ellipse in the plane of xz as its base.

It may be remarked that, in the case when ρ_1 and ρ_2 have the same sign, we might have taken for the osculating surface, instead of an ellipsoid, either an hyperboloid of two sheets or an elliptic paraboloid, both of which surfaces are convex as well as the ellipsoid. We might also, when ρ_2 is negative, ρ_1 being positive, have taken an hyperbolic paraboloid instead of an hyperboloid of one sheet, either of these surfaces being equally capable of representing both the magnitude and the direction of the curvature. The two surfaces, however, which have been selected out of the five, will be sufficient for the purpose of illustration.

(291) To speak generally, any two surfaces are said to *osculate* at a point where they have a common normal, when all normal sections made by the same plane have mutual osculation at the

point. That this may be the case, it is sufficient and necessary that the principal planes of the two surfaces coincide, and that their principal radii of curvature be equal and of the same nature. Suppose, in order to leave no obscurity on this point, that ρ, R, are the radii of the normal sections of the surfaces made by any normal plane. Let ε be the angle between the principal sections of the two surfaces. Then, ρ_1, ρ_2, being the principal radii of one surface, and R_1, R_2, of the other,

$$\frac{1}{\rho} = \frac{\cos^2 \alpha}{\rho_1} + \frac{\sin^2 \alpha}{\rho_2},$$

$$\frac{1}{R} = \frac{\cos^2 (\alpha + \varepsilon)}{R_1} + \frac{\sin^2 (\alpha + \varepsilon)}{R_2}.$$

But $R = \rho$ for all values of α; hence

$$\frac{\cos^2 \alpha}{\rho_1} + \frac{\sin^2 \alpha}{\rho_2} = \frac{\cos^2 (\alpha + \varepsilon)}{R_1} + \frac{\sin^2 (\alpha + \varepsilon)}{R_2}:$$

putting $\alpha = 0$, $\alpha = \frac{1}{2}\pi$, successively, we have

$$\frac{1}{\rho_1} = \frac{\cos^2 \varepsilon}{R_1} + \frac{\sin^2 \varepsilon}{R_2},$$

$$\frac{1}{\rho_2} = \frac{\sin^2 \varepsilon}{R_1} + \frac{\cos^2 \varepsilon}{R_2}.$$

and therefore

$$\frac{\cos^2 \alpha \cos^2 \varepsilon}{R_1} + \frac{\cos^2 \alpha \sin^2 \varepsilon}{R_2} + \frac{\sin^2 \alpha \sin^2 \varepsilon}{R_1} + \frac{\sin^2 \alpha \cos^2 \varepsilon}{R_2}$$

$$= \frac{\cos^2 (\alpha + \varepsilon)}{R_1} + \frac{\sin^2 (\alpha + \varepsilon)}{R_2};$$

hence $0 = \dfrac{2 \sin \alpha \cos \varepsilon \sin \varepsilon \cos \alpha}{R_2} - \dfrac{2 \cos \alpha \cos \varepsilon \sin \alpha \sin \varepsilon}{R_1}:$

since this is true for all values of α, we have

$$0 = \sin 2\varepsilon \left(\frac{1}{R_2} - \frac{1}{R_1}\right),$$

and therefore $\qquad \sin 2\varepsilon = 0,$

a relation which shews that the principal sections of the two surfaces coincide. Putting $\varepsilon = 0$, we see that

$$\frac{1}{\rho_1} = \frac{1}{R_1}, \quad \frac{1}{\rho_2} = \frac{1}{R_2},$$

or $\qquad \rho_1 = R_1, \quad \rho_2 = R_2.$

(292) *Umbilici.* The radii of curvature of the principal sections, and therefore of all normal sections, at an umbilicus are equal, and have the same signs. The conditions, therefore, for a point of this nature are that the values of $x - x'$, $y - y'$, $z - z'$, in (7), shall be the same both in magnitude and in sign for all simultaneous values of l, m, n, given by the equations (9) and (10). That this may be the case it is sufficient and necessary that

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' = C,$$

where C is invariable under the conditions of the problem. Now, from (9) and (10), we have

$$(mV + nW)^2 = l^2U^2,$$

$$2mnVW = l^2U^2 - m^2V^2 - n^2W^2;$$

similarly

$$2nlWU = m^2V^2 - n^2W^2 - l^2U^2,$$

$$2lmUV = n^2W^2 - l^2U^2 - m^2V^2:$$

combining these relations with the expression for C, we get

$$\begin{aligned} CUVW = UVW\,(l^2u + m^2v + n^2w) & \\ + Uu'\,(l^2U^2 - m^2V^2 - n^2W^2) & \\ + Vv'\,(m^2V^2 - n^2W^2 - l^2U^2) & \\ + Ww'\,(n^2W^2 - l^2U^2 - m^2V^2). & \end{aligned}$$

If we put $1 - m^2 - n^2$ in this equation for l^2, C will depend upon two quantities m^2 and n^2. Now, if between (9) and (10) we eliminate l, we shall have a quadratic for determining m in terms of n, such that for any arbitrary value of n^2, m^2 will have two different values; the converse being also, of course, true if we express n^2 in terms of m^2. Hence, in the expression for C in terms of m^2 and n^2, the coefficients of m^2 and n^2 must be each zero: this shews that, in the expression for C above given, the coefficients of l^2, m^2, n^2, must be equal. Hence

$$u + \frac{U}{VW}(Uu' - Vv' - Ww') = v + \frac{V}{WU}(Vv' - Ww' - Uu')$$

$$= w + \frac{W}{UV}(Ww' - Uu' - Vv') \ldots.(26).$$

We have tacitly supposed above, in assuming that for each value of m^2 there are two values of n^2, and vice versâ, that no

one of the quantities U, V, W, is zero: suppose, however, that V is zero, W either finite or zero, and that U is finite; then, from (9) and (10),

$$l = -n\frac{W}{U}, \quad m^2U^2 = U^2 - (U^2 + W^2)n^2;$$

and therefore

$$C = n^2\frac{W^2u}{U^2} + v - (U^2 + W^2)\frac{n^2v}{U^2}$$
$$+ n^2w + 2mnu' - 2n^2\frac{Wv'}{U} - 2mn\frac{Ww'}{U}.$$

Now for one value of n^2 there are two values of mn; hence the coefficient of mn must be zero: this being established, it is obvious also that the coefficient of n^2 must also be zero, n^2 being variable. Hence, when $V = 0$, we must have recourse to the conditions

$$Ww' = Uu', \quad W^2u + U^2w = (W^2 + U^2)v + 2WUv' \ldots (27).$$

If $W = 0$, the conditions for an umbilicus will, in like manner, be

$$Uu' = Vv', \quad U^2v + V^2u = (U^2 + V^2)w + 2UVw' \ldots (28);$$

and, if $U = 0$,

$$Vv' = Ww', \quad V^2w + W^2v = (V^2 + W^2)u + 2VWu' \ldots (29).$$

In order to ascertain the existence of umbilici on a surface, we must combine the equation to the surface with the equations (26), and determine whether these equations can be satisfied by three real simultaneous values of x, y, z, so as not to make U, V, W, any of them zero; the values of the co-ordinates will in this case determine an umbilicus. If the two equations (26) are equivalent to only one really independent equation, then this equation, together with that to the surface, will determine a certain curve on the surface of which every point is an umbilicus: such a line is called a *line of spherical curvature*, because at each of its points the surface possesses a uniform curvature like the sphere.

We must try also whether any one of the equations (27), (28), (29), can be satisfied, when V, W, or U, respectively, is zero; the equation (26) being in such cases inapplicable.

(293) We may readily transform our conditions for umbilici nto equivalent ones involving the partial differential coefficients

of z with respect to x and y. For this purpose we must replace $U, V, W, u, v, w, u', v', w'$, respectively, by

$$p, q, -1, r, t, 0, 0, 0, s.$$

Thus, (26) becomes

$$r - \frac{ps}{q} = t - \frac{qs}{p} = \frac{s}{pq},$$

or

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t},$$

which will be the proper relation, unless p or q be zero. If p or q be zero, we must have respectively, as the requisite conditions,

$$s = 0, \quad t = (1+q^2)\,r,$$

or

$$s = 0, \quad r = (1+p^2)\,t.$$

(294) In the case of the ellipsoid, its equation being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

we have

$$U = \frac{2x}{a^2}, \quad V = \frac{2y}{b^2}, \quad W = \frac{2z}{c^2},$$

$$u = \frac{2}{a^2}, \quad v = \frac{2}{b^2}, \quad w = \frac{2}{c^2},$$

$$u' = 0, \quad v' = 0, \quad w' = 0.$$

It is evident that the equations (26) are not satisfied. Suppose $V = 0$, so that $y = 0$; then, from (27),

$$\frac{z^2}{c^4a^2} + \frac{x^2}{a^4c^2} = \left(\frac{z^2}{c^4} + \frac{x^2}{a^4}\right)\frac{1}{b^2},$$

$$\frac{x^2}{a^2}(b^2 - c^2) = \frac{z^2}{c^2}(a^2 - b^2),$$

whence, from the equation to the ellipsoid,

$$x^2 = a^2\frac{a^2 - b^2}{a^2 - c^2}, \quad z^2 = c^2\frac{b^2 - c^2}{a^2 - c^2}.$$

From these relations it appears that there are four umbilici on the surface of an ellipsoid, situated in the principal section through its greatest and least axes; and that they are the positions of its vanishing circular sections.

(225) * The conditions for the existence of an umbilicus may likewise be deduced from the consideration that the two roots of the quadratic equation

$$\frac{U^2}{H-\dfrac{P}{\rho}}+\frac{V^2}{K-\dfrac{P}{\rho}}+\frac{W^2}{L-\dfrac{P}{\rho}}=0\ldots\ldots\ldots(a)$$

must be equal. Now if H, K, L, be not all equal when any values of x, y, z, are substituted, let K be the mean. It is readily seen that one root of the equation must lie between H and K, and the other between K and L. Hence, if the roots be equal, they must either be each equal to K, or we must have

$$H=K=L,$$

or

$$\left.\begin{aligned} &u+\frac{U}{VW}(Uu'-Vv'-Ww') \\ &=v+\frac{V}{WU}(Vv'-Ww'-Uu') \\ &=w+\frac{W}{UV}(Ww'-Uu'-Vv') \end{aligned}\right\}\ldots\ldots\ldots(b),$$

which are the general conditions for an umbilicus. The former condition, that each value of $\dfrac{P}{\rho}$ shall be equal to K, can only be satisfied if $V=0$. Now when any of the quantities U, V, or W, is equal to zero, the transformations of Art. (280), by which the products $2mn$, $2nl$, $2lm$, are eliminated, is impossible, or rather nugatory; and therefore, to find whether in any given surface there are umbilical points for which any of these quantities vanish, we should have recourse to the original expression for $\dfrac{P}{\rho}$. However, it is clear that if, making use of this expression, we had found by any other process an equation giving the required values of $\dfrac{P}{\rho}$ for any point of the surface, it would necessarily have been the same as (a) for general values of U, V, W, u, &c., though in a different form probably. Hence, if we can put (a) under a

* This method of investigating the conditions for umbilici was communicated to me by Mr. Thomson, of St. Peter's College, to whom I am also indebted for a knowledge of the formulæ themselves.

form which is not impossible or nugatory when any one of the quantities U, V, W, vanishes, it will give the values of $\frac{P}{\rho}$ at any point where such cases occur. Thus, let $U=0$. We may put (a) under the form

$$U^2\left(K-\frac{P}{\rho}\right)\left(L-\frac{P}{\rho}\right)$$
$$+\left(H-\frac{P}{\rho}\right)\left\{V^2\left(L-\frac{P}{\rho}\right)+W^2\left(K-\frac{P}{\rho}\right)\right\}=0.$$

Now we have from the values of H, K, L,

$$V^2L+W^2K=V^2w+W^2v-2VWu',$$

generally. Also, when $U=0$,

$$U^2K=0,\quad U^2L=0,\quad H=u,$$
$$U^2KL=-(Vv'-Ww')^2.$$

Hence the quadratic equation becomes, when $U=0$,

$$-(Vv'-Ww')^2+\left(u-\frac{P}{\rho}\right)\left\{V^2w+W^2v-2VWu'-(V^2+W^2)\frac{P}{\rho}\right\}=0,$$

or $$\left(\frac{P}{\rho}-u\right)\left(\frac{P}{\rho}-\frac{V^2w+W^2v-2VWu'}{V^2+W^2}\right)=\frac{(Vv'-Ww')^2}{V^2+W^2}.$$

This must be considered as supplementary to equation (a), giving the value of $\frac{P}{\rho}$ for any point where $U=0$. Symmetrical equations apply to the cases where $V=0$, or $W=0$. Now the condition that any equation of the form

$$(z-a)(z-b)=c^2$$

may have equal roots, is

$$(a+b)^2=4(ab-c^2),$$

which requires that

$$c=0 \text{ and } a=b.$$

Hence the conditions for an umbilicus, where

$$\left.\begin{aligned} &U=0,\\ \text{are}\qquad &Vv'-Ww'=0,\\ \text{and}\qquad &u=\frac{V^2w+W^2v-2VWu'}{V^2+W^2}\end{aligned}\right\}\ldots\ldots\ldots\ldots(c).$$

Similarly the conditions for an umbilicus, where

$$V = 0,$$

are

$$\left.\begin{aligned} &Ww' - Uu' = 0 \\ &v = \frac{W^2u + U^2w - 2WUv'}{W^2 + U^2} \end{aligned}\right\} \dots\dots\dots (d):$$

and

and, where

$$W = 0,$$

are

$$\left.\begin{aligned} &Uu' - Vv' = 0 \\ &w = \frac{U^2v + V^2u - 2UVw'}{U^2 + V^2} \end{aligned}\right\} \dots\dots\dots\dots (e).$$

Hence, to find all the umbilici of a surface, we must first satisfy the general conditions (b) for as many points as possible, and we must then try whether there are any points for which any one of the special systems (c), (d), (e), is satisfied.

(296) In the preceding Article it has been remarked that, although in certain cases the investigation of Art. (280) involves indeterminate operations, yet the equation (18) there obtained will be universally true, and that it must be essentially the same as if it had been found by a process involving no nugatory expressions. * We shall, however, now give an investigation which is at all times free from nugatory operations, in order that every light may be thrown upon this important equation.

We have to render

$$\frac{P}{\rho} = l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' \dots . (a)$$

a maximum or minimum, subject to the conditions

$$lU + mV + nW = 0 \dots\dots\dots (b),$$

$$l^2 + m^2 + n^2 = 1 \dots\dots\dots (c).$$

Differentiating (a), (b), (c), and putting $d\rho = 0$, we get

$$0 = (lu + mw' + nv')\, dl + (mv + nu' + lw')\, dm + (nw + lv' + mu')\, dn,$$

$$0 = ldl + mdm + ndn,$$

$$0 = Udl + Vdm + Wdn.$$

* This investigation was given by Mr. Greatheed, of Trinity College, in the *Cambridge Mathematical Journal*, for May, 1838.

From which equations we obtain, by indeterminate multipliers,

$$lu + mw' + nv' = \lambda l + \mu U,$$
$$mv + nu' + lw' = \lambda m + \mu V,$$
$$nw + lv' + mu' = \lambda n + \mu W.$$

Multiplying these equations by l, m, n, respectively, and adding, we have, by the aid of (a), (b), (c),

$$\lambda = \frac{P}{\rho}.$$

Hence
$$\left(u - \frac{P}{\rho}\right) l + w'm + v'n = \mu U,$$
$$\left(v - \frac{P}{\rho}\right) m + u'n + w'l = \mu V,$$
$$\left(w - \frac{P}{\rho}\right) n + v'l + u'm = \mu W.$$

If from these three equations we eliminate m and n by cross multiplication, we see that the coefficient of l is symmetrical with respect to the coefficients; so that if we call it S, and write

$$l = \frac{\mu}{S}\left[U\left\{\left(v - \frac{P}{\rho}\right)\left(w - \frac{P}{\rho}\right) - u'^2\right\} + V\left\{u'v' - w'\left(w - \frac{P}{\rho}\right)\right\} + W\left\{w'u' - v'\left(v - \frac{P}{\rho}\right)\right\}\right],$$

we shall have also

$$m = \frac{\mu}{S}\left[V\left\{\left(w - \frac{P}{\rho}\right)\left(u - \frac{P}{\rho}\right) - v'^2\right\} + W\left\{v'w' - u'\left(u - \frac{P}{\rho}\right)\right\} + U\left\{u'v' - w'\left(w - \frac{P}{\rho}\right)\right\}\right],$$

$$n = \frac{\mu}{S}\left[W\left\{\left(u - \frac{P}{\rho}\right)\left(v - \frac{P}{\rho}\right) - w'^2\right\} + U\left\{w'u' - v'\left(v - \frac{P}{\rho}\right)\right\} + V\left\{v'w' - u'\left(u - \frac{P}{\rho}\right)\right\}\right].$$

If we multiply these three equations by U, V, W, respectively, and add, we have, by the relation (b),

$$U^2\left(v - \frac{P}{\rho}\right)\left(w - \frac{P}{\rho}\right) + V^2\left(w - \frac{P}{\rho}\right)\left(u - \frac{P}{\rho}\right) + W^2\left(u - \frac{P}{\rho}\right)\left(v - \frac{P}{\rho}\right)$$
$$- 2VWu'\left(u - \frac{P}{\rho}\right) - 2WUv'\left(v - \frac{P}{\rho}\right) - 2UVw'\left(w - \frac{P}{\rho}\right)$$
$$- U^2u'^2 - V^2v'^2 - W^2w'^2 + 2VWv'w' + 2WUw'u' + 2UVu'v' = 0.$$

This is a quadratic equation in $\frac{P}{\rho}$, with which the equation (18) will coincide when cleared of its fractional form.

(297) Having considered above the properties of the normal curvature of surfaces, we will now establish an important theorem, due to Meunier, by which the curvature of an oblique section at any point of a surface may be immediately obtained from that of a normal section at the same point.

Differentiating the equation to the surface, we have

$$U\,dx + V\,dy + W\,dz = 0,$$

$$\begin{aligned} U\,d^2x + V\,d^2y + W\,d^2z + u\,dx^2 + v\,dy^2 + w\,dz^2 \\ + 2u'\,dy\,dz + 2v'\,dz\,dx \\ + 2w'\,dx\,dy = 0. \end{aligned}$$

Let R be the radius of curvature of the oblique section; then, by the theory of the curvature of curves in space (Art. 265), we know that, α, β, γ, being the angles which R makes with the co-ordinate axes,

$$d^2x = \frac{ds^2}{R}\cos\alpha,$$

$$d^2y = \frac{ds^2}{R}\cos\beta,$$

$$d^2z = \frac{ds^2}{R}\cos\gamma.$$

Hence we get

$$0 = \frac{1}{R}(U\cos\alpha + V\cos\beta + W\cos\gamma) + \&c.:$$

but, θ denoting the inclination of the section to the normal plane,

$$\cos\theta = \frac{U\cos\alpha + V\cos\beta + W\cos\gamma}{\pm(U^2 + V^2 + W^2)^{\frac{1}{2}}};$$

hence

$$0 = \frac{\cos\theta}{R} + \&c.$$

It will be observed, that the remaining terms of the equation are functions only of the partial differential coefficients of $F(x, y, z)$, and of $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$; these terms will therefore remain constant so long as the oblique section passes through

the same tangent line to the surface at the point under consideration. Hence, under this condition,

$$\frac{\cos\theta}{R} = \text{constant}.$$

When $\theta = 0$, $R = \rho$, ρ being the radius of curvature of the normal section: hence $$R = \rho\cos\theta,$$

or the radius of curvature of an oblique section is the projection, upon the plane of this curve, of the radius of curvature of the normal section which passes through the same tangent line.

(298) A line of curvature in any surface is the locus of a series of its consecutive points, such that the normals at each point shall meet the normal at the consecutive one.

The equations to the normal are

$$\frac{x'-x}{U} = \frac{y'-y}{V} = \frac{z'-z}{W}.$$

Let each member of this equation be represented by Q. Then

$$x' = x + QU,$$
$$y' = y + QV,$$
$$z' = z + QW.$$

If the consecutive normals meet, x', y', z', will remain constant when x, y, z, and therefore U, V, W, Q, vary infinitesimally. Hence we have

$$dx + QdU + UdQ = 0,$$
$$dy + QdV + VdQ = 0,$$
$$dz + QdW + WdQ = 0:$$

or, putting $$\frac{dx}{ds} = l,\quad \frac{dy}{ds} = m,\quad \frac{dz}{ds} = n,$$

$$\left.\begin{aligned} lu + mw' + nv' + \frac{l}{Q} + \frac{U}{Q}\cdot\frac{dQ}{ds} &= 0,\\ mv + nu' + lw' + \frac{m}{Q} + \frac{V}{Q}\cdot\frac{dQ}{ds} &= 0,\\ nw + lv' + mu' + \frac{n}{Q} + \frac{W}{Q}\cdot\frac{dQ}{ds} &= 0,\end{aligned}\right\} \quad \ldots\ldots (30).$$

Eliminating Q and dQ from our equations, we find

$$U(dVdz-dWdy)+V(dWdx-dUdz)+W(dUdy-dVdx)=0,$$

as the differential equation for the lines of curvature. This equation, together with the equation to the surface, involves all the properties of lines of curvature. The process of integration, the differential equation being of two dimensions in dx, dy, dz, will introduce into our results an arbitrary constant so involved that, when x, y, z, are given, it will have two different values. If we substitute these two values of the constant in the integral, we shall thus get the equations to two lines of curvature passing through the given point.

(299) Differentiating the equations (9), (10), (13), considering l, m, n, variable, and putting $d\rho=0$, which corresponds to the determination of the principal sections, we get

$$ldl + mdm + ndn = 0,$$
$$Udl + Vdm + Wdn = 0,$$
$$(lu+mw'+nv')\,dl+(mv+nu'+lw')\,dm+(nw+lv'+mu')\,dn=0.$$

Multiplying these equations in order by λ, μ, 1, adding and equating to zero the coefficients of dl, dm, dn, we obtain

$$lu + mw' + nv' = \lambda l + \mu U,$$
$$mv + nu' + lw' = \lambda m + \mu V,$$
$$nw + lv' + mu' = \lambda n + \mu W.$$

Now, if between these three equations λ and μ be eliminated, it is clear that we shall get an equation in l, m, n coinciding with the equation resulting from the elimination of Q and dQ between the equations (30). From this it is evident that, l, m, n, being in both cases subject to the same equations (9) and (10), the directions in which the lines of curvature start from any point on the surface coincide with the directions of greatest and least curvature, and are therefore at right angles to each other.

(300) That the two lines of curvature through any point of a surface are at right angles to one another, may be demonstrated also in the following manner.*

* For this demonstration of the perpendicularity of the lines of curvature, I am indebted to Mr. Fischer, of Pembroke College.

We have for the differential equation of the lines of curvature,

$$(VdW - WdV)\,dx + (WdU - UdW)\,dy + (UdV - VdU)\,dz = 0.$$

Now $\quad VdW - WdV = (Vv' - Ww')\,dx + (Vu' - Wv)\,dy$
$$+ (Vw - Wu')\,dz:$$

also $$Udx + Vdy + Wdz = 0.$$

Therefore, eliminating dx,

$$VdW - WdV = \left\{Vu' - Wv - \frac{V}{U}(Vv' - Ww')\right\} dy$$
$$+ \left\{Vw - Wu' - \frac{W}{U}(Vv' - Ww')\right\} dz$$
$$= -WKdy + VLdz.$$

Modifying the second and third terms of the above equation similarly, and arranging the result, we find

$$U(K - L)\,dy\,dz + V(L - H)\,dz\,dx + W(H - K)\,dx\,dy = 0.$$

Hence, if l, m, n, be the direction-cosines of the tangent to a line of curvature through any point of the surface, their ratios are determined by the equations

$$Ul + Vm + Wn = 0 \;\ldots\ldots\ldots\; (a),$$
$$U(K - L)\,mn + V(L - H)\,nl + W(H - K)\,lm = 0 \;\ldots\; (b).$$

It is readily shewn that two, and only two, systems of values of the ratios may be deduced from these equations, and that they are all real. Hence, if l_1, m_1, n_1, and l_2, m_2, n_2, be the values, we may write down equation (a) for each system; and we thence deduce, by a common process,

$$\frac{U}{m_1n_2 - m_2n_1} = \frac{V}{n_1l_2 - n_2l_1} = \frac{W}{l_1m_2 - l_2m_1} \;\ldots\ldots\; (c).$$

Similarly, from (b) we deduce

$$\frac{U(K - L)}{l_1l_2\,(m_1n_2 - m_2n_1)} = \frac{V(L - H)}{m_1m_2\,(n_1l_2 - n_2l_1)} = \frac{W(H - K)}{n_1n_2\,(l_1m_2 - l_2m_1)} \;\ldots\; (d).$$

Therefore, dividing the members of equations (c) by the corresponding members of (d), we have

$$\frac{l_1l_2}{K - L} = \frac{m_1m_2}{L - H} = \frac{n_1n_2}{H - K} = \frac{l_1l_2 + m_1m_2 + n_1n_2}{0};$$

therefore $$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

Hence the two directions of the lines of curvature through any point are at right angles.

(301) The identity of the equations for finding the positions of the principal sections and the directions of the lines of curvature at any point of a surface, points out to us a method of finding at once the principal sections and principal radii at any point of a surface of revolution. Let P be any point of the surface; M the point where the axis of revolution is intersected by a perpendicular let fall upon it from P, N being the intersection of the axis with a normal to the generating curve at P. The plane of the generating curve will evidently be one of the principal sections, since any two of its consecutive normals must meet; and the other principal section must pass through the normal PN at right angles to the area of the generating curve. The principal radius of curvature of the former section will be the radius of curvature of the generating curve. The radius of curvature of the latter section will, by Meunier's theorem, be equal to the product of PM, which is the radius of the circular section of the surface through P, and the secant of the angle MPN; that is, it will be the line PN.

(302) To find the lines of curvature on the surface of an ellipsoid.

The equation to the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \ldots\ldots\ldots\ldots (1)';$$

whence also

$$\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2} + \frac{z\,dz}{c^2} = 0 \ldots\ldots\ldots (2)'.$$

Also, from the differential equation for the lines of curvature we have, in the case of the ellipsoid,

$$(b^2 - c^2)\frac{x}{dx} + (c^2 - a^2)\frac{y}{dy} + (a^2 - b^2)\frac{z}{dz} = 0 \ldots (3)'.$$

Assume

$$A\frac{x^2}{a^2} + B\frac{y^2}{b^2} + C\frac{z^2}{c^2} = 0 \quad \ldots\ldots\ldots\ldots (4)'$$

to be an equation between x, y, z, which, together with (1)', shall determine the lines of curvature. The admissibility of this assumption depends upon the possibility of its satisfying (3)' in such a way that the three constants A, B, C, shall

be equivalent to a constant with two values. That these constants should be subject to this condition will be clear when we consider, that if between (1)′, (2)′, and (3)′, we eliminate z and dz, we shall have a differential equation of the first order and second degree in x and y.

From (4)′ there is

$$A\frac{x\,dx}{a^2}+B\frac{y\,dy}{b^2}+C\frac{z\,dz}{c^2}=0,$$

whence, by the aid of (2)′, we see that

$$\frac{x\,dx}{a^2}:\frac{y\,dy}{b^2}:\frac{z\,dz}{c^2}::B-C:C-A:A-B\ldots.(5)'.$$

From (3)′ and (5)′ we get

$$(b^2-c^2)\frac{x^2}{a^2(B-C)}+(c^2-a^2)\frac{y^2}{b^2(C-A)}+(a^2-b^2)\frac{z^2}{c^2(A-B)}=0.$$

This equation will coincide with (4)′, provided that

$$A=\lambda\frac{b^2-c^2}{B-C},\quad B=\lambda\frac{c^2-a^2}{C-A},\quad C=\lambda\frac{a^2-b^2}{A-B},$$

where λ is any quantity whatever: these three equations establish only one relation between A, B, C, viz.

$$\frac{b^2-c^2}{A}+\frac{c^2-a^2}{B}+\frac{a^2-b^2}{C}=0\ldots\ldots(6)'.$$

If we put $A=\dfrac{b^2-c^2}{f},\quad B=\dfrac{c^2-a^2}{g},\quad C=\dfrac{a^2-b^2}{h},$

then $$f+g+h=0\ldots\ldots\ldots\ldots(7)',$$

$$\frac{b^2-c^2}{f}\frac{x^2}{a^2}+\frac{c^2-a^2}{g}\frac{y^2}{b^2}+\frac{a^2-b^2}{h}\frac{z^2}{c^2}=0\ldots.(8)'.$$

Equation (8)′, together with (1)′, are the equations to the lines of curvature. If we substitute in (8)′ the quantity $-(f+g)$ for h, by virtue of (7)′, the resulting equation will involve only one arbitrary constant, viz. $\dfrac{g}{f}$. In order to determine the value of this constant, let us suppose that (x_1, y_1, z_1) is a point on the surface of the ellipsoid from which a conjugate pair of lines of curvature start.

Then, from (8)′ we have

$$\frac{b^2 - c^2}{f}\frac{x_i^2}{a^2} + \frac{c^2 - a^2}{g}\frac{y_i^2}{b^2} = -\frac{a^2 - b^2}{h}\frac{z_i^2}{c^2};$$

and from (7)′ there is $\quad f + g = -h:$

multiplying these two equations together we get

$$\frac{g}{f}(b^2 - c^2)\frac{x_i^2}{a^2} + \frac{f}{g}(c^2 - a^2)\frac{y_i^2}{b^2} + (b^2 - c^2)\frac{x_i^2}{a^2} + (c^2 - a^2)\frac{y_i^2}{b^2}$$
$$= (a^2 - b^2)\frac{z_i^2}{c^2} \dots . (9)',$$

which is a quadratic in $\frac{g}{f}$, of which the roots, as may be readily ascertained, have real values. The double value of the constant $\frac{g}{f}$ satisfies one of the two conditions on which the legitimacy of the assumption (4)′ is dependent.

The equation (8)′ belongs to a cone of the second order, with its vertex at the origin, which shews that the lines of curvature through any point of an ellipsoid are the intersections of the surface with two cones, of which the vertices are both at the centre: the fact of there being two cones is shewn by the double value of the constant $\frac{g}{f}$.

The equations (5)′, putting for A, B, C, the values obtained above, become

$$\frac{x\,dx}{a^2\left(\frac{c^2 - a^2}{g} - \frac{a^2 - b^2}{h}\right)} = \frac{y\,dy}{b^2\left(\frac{a^2 - b^2}{h} - \frac{b^2 - c^2}{f}\right)} = \frac{z\,dz}{c^2\left(\frac{b^2 - c^2}{f} - \frac{c^2 - a^2}{g}\right)},$$

which are the differential equations of the projections of the lines of curvature on the co-ordinate planes: these projections, as the forms of their differential equations shew, are conic sections.

Suppose that

$$x_i^2 = a^2\frac{a^2 - b^2}{a^2 - c^2}, \quad y_i^2 = \beta^2, \quad z_i^2 = c^2\frac{b^2 - c^2}{a^2 - c^2},$$

β being a very small quantity. Then, from (9)′ we have

$$(b^2 - c^2)\frac{x_i^2}{a^2}\cdot\frac{g}{f} + (c^2 - a^2)\frac{\beta^2}{b^2\cdot\frac{g}{f}} + (c^2 - a^2)\frac{\beta^2}{b^2} = 0;$$

from which equation it is evident that, as β diminishes indefinitely without absolutely vanishing, the values of $\frac{g}{f}$ approach indefinitely nearly to zero. We may see from (7)′ that the corresponding values of $\frac{g}{h}$ are indefinitely near to zero. From this it is evident that the equation (8)′, as the point (x_1, y_1, z_1) approaches indefinitely near to an umbilicus, without absolutely coinciding with it, degenerates indefinitely nearly into

$$y^2 = 0;$$

or that, to proceed to the limit, the lines of curvature through an umbilicus coincide with the section of the ellipsoid made by a plane through its greatest and least axes.

If in the equations (2)′ and (3)′ we put $y = 0$, they become

$$\frac{xdx}{a^2} = -\frac{zdz}{c^2}$$

$$(b^2 - c^2)\frac{x}{dx} = -(a^2 - b^2)\frac{z}{dz};$$

whence
$$\frac{b^2 - c^2}{a^2}x^2 = \frac{a^2 - b^2}{c^2}z^2,$$

a relation which is satisfied by the values of x, z, at an umbilicus without subjecting to any restriction the ratios between the differentials dx, dy, dz. This shews that the property of the intersections of consecutive normals, as far as the first order of differentials is concerned, is satisfied equally in whatever direction an indefinitely small arc is taken on the surface of the ellipsoid starting from an umbilicus.

This result may at first sight appear to be incompatible with our former conclusion, viz. that the lines of curvature through an umbilicus, coincide with the plane of xz. Conceive, however, an indefinitely small ring to be described on the surface around the umbilicus: from each point of this ring a pair of lines of curvature will start indefinitely nearly coincident with the plane of xz. Hence we see, that in whatever direction we may start from the umbilicus to this infinitesimal ring, the

subsequent course of the lines of curvature will be ultimately the same.

The discovery that the lines of curvature on an ellipsoid are the intersections of the surface with cones of the second order, is due to Mr. Leslie Ellis: the more direct investigation by which he arrived at this conclusion may be seen in the *Cambridge Mathematical Journal* for May 1840. The student is recommended also to consult Leroy's *Géométrie Descriptive*, for graphic illustrations of the forms of the lines of curvature.

(303) If there be three series of surfaces, such that all the surfaces of each series cut the surfaces of the other two series at right angles, the lines of intersection of any one of the surfaces of the three series, with the surfaces of the two conjugate series, are its lines of curvature. This remarkable theorem was given by Dupin, in his *Développements de Géométrie, Cinquième Mémoire.*

* Let O be any point in which three conjugate surfaces intersect, and let the rectangular axes OX, OY, OZ, be perpendicular to the tangent planes of the three surfaces at O. Let

$$F(x, y, z) = \lambda \dots\dots\dots\dots\dots (a),$$
$$F_1(x, y, z) = \lambda_1 \dots\dots\dots\dots\dots (a_1),$$
$$F_2(x, y, z) = \lambda_2 \dots\dots\dots\dots\dots (a_2),$$

be the equations of the three series; and, when proper values are attached to λ, λ_1, λ_2, let (a) be the surface touched by YOZ, (a_1) by ZOX, and (a_2) by XOY. Hence, when $x = 0$, $y = 0$, $z = 0$, we have

$$\left.\begin{array}{ll} V = 0, & W = 0, \\ W_1 = 0, & U_1 = 0, \\ U_2 = 0, & V_2 = 0, \end{array}\right\} \dots\dots\dots\dots (b),$$

the suffixes of the letters in (b) connecting them with the corresponding surfaces.

Now, since the system is orthogonal, we must have identically

$$\left.\begin{array}{r} U_1U_2 + V_1V_2 + W_1W_2 = 0 \\ U_2U + V_2V + W_2W = 0 \\ UU_1 + VV_1 + WW_1 = 0 \end{array}\right\} \dots\dots\dots\dots (c).$$

* This demonstration of Dupin's theorem was given by Mr. Thomson, of St. Peter's College, in the *Cambridge Mathematical Journal* for February 1844.

Differentiating the first of these equations with respect to x, the second with respect to y, and the third with respect to z, putting x, y, z, each equal to zero, in the result, and making use of equations (b), we have

$$(V_1)(w_2') + (W_2)(v_1') = 0,$$
$$(W_2)(u') + (U)(w_2') = 0,$$
$$(U)(v_1') + (V_1)(u') = 0,$$

the brackets denoting that, in the quantities enclosed, x, y, z, are equated to zero. From these equations we conclude that

$$(u') = 0, \quad (v_1') = 0, \quad (w_2') = 0.$$

The relation $(u') = 0$, see Arts. (284, 285), shews that the planes of xy and xz contain the principal sections of (a) through O, and therefore the lines of intersection of (a_1) and (a_2) with (a) touch the principal sections of (a) at O. Now O may be any point in one of the surfaces (a), and therefore each of these surfaces has its lines of curvature traced upon it by the surfaces of the series (a_1), (a_2). Similarly, from the equations $(v_1') = 0$, $(w_2') = 0$, it follows that each surface (a_1) has its lines of curvature traced by (a_2) and (a), and each surface (a_2) by (a) and (a_1), which is the theorem to be proved.

It will be observed that in this demonstration only one surface of each series has been considered. Hence the theorem proved is, that if any three surfaces cut one another at right angles along each line of intersection, at any point where all three meet, the lines of intersection on each surface will be tangents to its principal sections. Dupin's theorem follows immediately from this result.

CHAPTER XVI.

PROBLEMS.

PROB. 1. To find the relations between the co-ordinates of the extremities of three conjugate diameters in an ellipsoid.

Let the equation to the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This is satisfied by

$$x = al, \quad y = bm, \quad z = cn,$$

provided that $\qquad l^2 + m^2 + n^2 = 1 \;\ldots\ldots\ldots\ldots\ldots (1),$

and therefore al, bm, cn, may be taken as co-ordinates of the extremity of a semi-diameter r.

In like manner we may take

$$x' = al', \quad y' = bm', \quad z' = cn',$$

under the condition $\qquad l'^2 + m'^2 + n'^2 = 1 \;\ldots\ldots\ldots\ldots\ldots (2),$

and $\qquad x'' = al'', \quad y'' = bm'', \quad z'' = cn'',$

under the condition $\qquad l''^2 + m''^2 + n''^2 = 1 \;\ldots\ldots\ldots\ldots\ldots (3),$

as the co-ordinates of the extremity of two other semi-diameters r' and r''.

Now if we change the co-ordinate axes so as to coincide with r, r', and r'', we shall have to put

$$x = \frac{al}{r} x_1 + \frac{al'}{r'} y_1 + \frac{al''}{r''} z_1,$$

and similarly for the others. If we substitute these values in the equation to the surface, and make the conditions that r, r', r'', shall be conjugate semi-diameters, which involves the vanishing of

the terms containing the rectangles in the transformed equation, we find

$$\left.\begin{array}{r} ll' + mm' + nn' = 0 \\ l'l'' + m'm'' + n'n'' = 0 \\ l''l + m''m + n''n = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(4).$$

The equations (4) shew that

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0,$$

$$\frac{x'x''}{a^2} + \frac{y'y''}{b^2} + \frac{z'z''}{c^2} = 0,$$

$$\frac{x''x}{a^2} + \frac{y''y}{b^2} + \frac{z''z}{c^2} = 0,$$

which are the required relations between the co-ordinates (x, y, z), (x', y', z'), (x'', y'', z''), of the extremities of three conjugate diameters.

COR. 1. From the equations (4), we see that (l, m, n), (l', m', n'), (l'', m'', n''), are the direction-cosines of three lines which are at right angles to each other.

Cor. 2. Let (X, Y, Z), (X', Y', Z'), (X'', Y'', Z''), be three such points and R such a line that

$$\begin{array}{lll} Rl = X, & Rm = Y, & Rn = Z, \\ Rl' = X', & Rm' = Y', & Rn' = Z', \\ Rl'' = X'', & Rm'' = Y'', & Rn'' = Z''; \end{array}$$

then, from the above conclusions, we easily see that if the points

$$(X, Y, Z), (X', Y', Z'), (X'', Y'', Z''),$$

be the extremities of three radii, at right angles to each other, of a sphere

$$x^2 + y^2 + z^2 = R^2,$$

the points

$$\left(\frac{aX}{R}, \frac{bY}{R}, \frac{cZ}{R}\right), \left(\frac{aX'}{R}, \frac{bY'}{R}, \frac{cZ'}{R}\right), \left(\frac{aX''}{R}, \frac{bY''}{R}, \frac{cZ''}{R}\right),$$

will be the extremities of conjugate diameters of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

For further information on this subject the reader is referred to a memoir by M. Brassine, in Liouville's *Journal de Mathématiques*, *Av.* 1842.

PROB. 2. The sum of the squares of the projections of any three conjugate diameters on a fixed line is constant.

Instead of projecting the diameters on the line directly, it is better to project the co-ordinates of the extremities of each diameter, and add them. Now if λ, μ, ν, be the direction-cosines of the given line, the sum of the projections of the co-ordinates of the extremity of one diameter is

$$al\lambda + bm\mu + cn\nu:$$

similarly, for the other two, we have

$$al'\lambda + bm'\mu + cn'\nu,$$

$$al''\lambda + bm''\mu + cn''\nu.$$

Squaring and adding, and observing that both the axes of co-ordinates and the lines of which the direction cosines are (l, m, n), (l', m', n'), (l'', m'', n''), are rectangular systems, we shall have for the required sum,

$$a^2\lambda^2 + b^2\mu^2 + c^2\nu^2,$$

which is a constant quantity.

PROB. 3. The sum of the squares of the perpendiculars drawn from the extremities of three conjugate diameters on a fixed diametral plane is constant.

If the equation to the plane be

$$\lambda x + \mu y + \nu z = 0,$$

in which λ, μ, ν, are the direction-cosines, and if p, p', p'', be three conjugate perpendiculars,

$$p = al\lambda + bm\mu + cn\nu,$$

$$p' = al'\lambda + bm'\mu + cn'\nu,$$

$$p'' = al''\lambda + bm''\mu + cn''\nu,$$

and therefore

$$p^2 + p'^2 + p''^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2, \text{ a constant.}$$

PROB. 4. The sum of the squares of the reciprocals of three diameters of an ellipsoid at right angles to each other is constant.

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and r being any diameter of which the direction-cosines are $l, m, n,$

$$x = rl, \quad y = rm, \quad z = rn;$$

therefore

$$\frac{1}{r^2} = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}.$$

Similarly for another diameter,

$$\frac{1}{r_1^2} = \frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}:$$

and again

$$\frac{1}{r_2^2} = \frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}.$$

Adding, then, and observing that, in consequence of the diameters being at right angles to each other,

$$l^2 + l_1^2 + l_2^2 = 1, \quad m^2 + m_1^2 + m_2^2 = 1, \quad n^2 + n_1^2 + n_2^2 = 1,$$

we have

$$\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

PROB. 5. To find the locus of the centres of the sections in a central surface of the second order made by planes which all pass through one point.

Let the equation to the surface be

$$Ax^2 + By^2 + Cz^2 = 1 \ldots\ldots\ldots\ldots\ldots (1),$$

$a, b, c,$ the co-ordinates of the fixed point through which all the planes pass. The equation to any one of them must be of the form $\quad l(x - a) + m(y - b) + n(z - c) = 0 \ldots\ldots\ldots (2).$

Now the equation to the line which is the locus of the centres of planes parallel to this one are

$$\frac{Ax}{l} = \frac{By}{m} = \frac{Cz}{n} \ldots\ldots\ldots\ldots\ldots (3).$$

Therefore the co-ordinates of the centre of the section must satisfy equations (2) and (3). Eliminating $l, m, n,$ between them, we have

$$Ax(x - a) + By(y - b) + Cz(z - c) = 0$$

as the equation to the required locus, which is evidently a surface similar to (1), and passing through the origin and the given point.

PROB. 6. To find the locus of the intersection of tangent planes to an ellipsoid drawn at the extremities of a system of conjugate diameters.

Let the co-ordinates of the extremities be $x_0\, y_0\, z_0$, $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$; then the equations to the tangent planes are

$$Ax_0x + A'y_0y + A''z_0z = 1,$$
$$Ax_1x + A'y_1y + A''z_1z = 1,$$
$$Ax_2x + A'y_2y + A''z_2z = 1.$$

Adding, then, we have

$$Ax(x_0 + x_1 + x_2) + A'y(y_0 + y_1 + y_2) + A''z(z_0 + z_1 + z_2) = 3.$$

Now, as the diameters are conjugate, the tangent plane at the extremity of any one is parallel to the diametral plane containing the other two, and therefore the point of intersection (xyz) is the extremity of the diagonal of a parallelopiped of which the three diameters are conterminous edges. Now the projection of the diagonal on any line is equal to the sum of the projections on the same line of the three edges terminated at one extremity of the diagonal. Hence, making the three axes in turn the lines of projection, we have

$$x = x_0 + x_1 + x_2, \quad y = y_0 + y_1 + y_2, \quad z = z_0 + z_1 + z_2.$$

Substituting in the preceding equation, it becomes

$$Ax^2 + A'y^2 + A''z^2 = 3,$$

which is the equation to the required locus. It is obvious that this is an ellipsoid concentric with and similar to the original one, and that its axes are greater in the ratio of $\sqrt{3}$ to 1.

PROB. 7. If at a point P in a curved surface a tangent plane be drawn, on which a perpendicular OY be drawn from a fixed point O; and if in OY a point P' be taken such that $OP'.OY=k^2$ (a constant), the locus of P' will be a surface such that the perpendicular from O on its tangent plane at P' passes through P, and if the length of this perpendicular be OY', there exists the relation $OP.OY' = k^2$.

Let the co-ordinates, measured from O, of P be (xyz), those of P' $(x'y'z')$; then, if ϕ be the angle between OP and OY,

$$k^2 = OP'.OY = OP'.OP.\cos\phi = OP'.OP.\,\frac{xx' + yy' + zz'}{OP'.OP}$$
$$= xx' + yy' + zz' \ldots . (1).$$

Now if $$F(x', y', z') = 0 \ldots\ldots\ldots\ldots\ldots (2)$$
be the equation to the locus of P', and if p_1 be the perpendicular from O on its tangent plane,

$$p_1 = \frac{x'\frac{dF}{dx'} + y'\frac{dF}{dy'} + z'\frac{dF}{dz'}}{\left\{\left(\frac{dF}{dx'}\right)^2 + \left(\frac{dF}{dy'}\right)^2 + \left(\frac{dF}{dz'}\right)^2\right\}^{\frac{1}{2}}} \ldots\ldots\ldots (3).$$

Differentiating (1) and (2), considering xyz, $x'y'z'$, as all variable, we have

$$x'dx + y'dy + z'dz + xdx' + ydy' + zdz' = 0 \ldots\ldots (4),$$

$$\frac{dF}{dx'}dx' + \frac{dF}{dy'}dy' + \frac{dF}{dz'}dz' = 0 \ldots\ldots\ldots\ldots (5);$$

to which, if $f(x, y, z) = 0$ be the equation to the given surface, we have to join

$$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0 \ldots\ldots\ldots\ldots (6);$$

$\lambda(5) + \mu(6) - (4)$ gives, on equating to zero the coefficients of each differential,

$$x = \lambda\frac{dF}{dx'}, \quad y = \lambda\frac{dF}{dy'}, \quad z = \lambda\frac{dF}{dz'} \ldots\ldots (7),$$

$$x' = \mu\frac{df}{dx}, \quad y' = \mu\frac{df}{dy}, \quad z' = \mu\frac{df}{dz} \ldots\ldots (8).$$

The equations (7) indicate that the line of which the direction-cosines are proportional to x, y, z, coincides with that of which the direction-cosines are proportional to $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$; that is, that OP and OY' coincide, or that OY' passes through P. Equations (8) merely indicate the original construction. Also from (3) and (7), we have

$$p_1 = \frac{xx' + yy' + zz'}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{k^2}{r},$$

or $$p_1 r = k^2, \text{ that is, } OP.OY' = k^2.$$

Surfaces related to each other in the manner described above are called *reciprocal* surfaces by Professor Maccullagh, who, in

the *Transactions of the Royal Irish Academy*, vol. XVII., has investigated many of their properties, and applied them most ingeniously to researches in the Theory of Light.

If the original surface be an ellipsoid, the reciprocal surface will also be one—such that the products of the corresponding axes are equal.

PROB. 8. A straight line moves so as to have three of its points constantly in three fixed planes; to find the surface traced out by any other point.

Let the equations to the fixed planes be

$$lx + my + nz = 0, \quad l'x + m'y + n'z = 0, \quad l''x + m''y + n''z = 0.$$

Let $a, b, c,$ be the distances of any assumed point (xyz) in the line from the three points which are to rest in the three planes. If the co-ordinates of these points be $\alpha, \beta, \gamma;\ \alpha', \beta', \gamma';\ \alpha'', \beta'', \gamma'';$ and the direction-cosines be $\lambda, \mu, \nu,$ the equations to the line may be put in three forms

$$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu} = a,$$

$$\frac{x-\alpha'}{\lambda} = \frac{y-\beta'}{\mu} = \frac{z-\gamma'}{\nu} = b,$$

$$\frac{x-\alpha''}{\lambda} = \frac{y-\beta''}{\mu} = \frac{z-\gamma''}{\nu} = c.$$

But the co-ordinates $\alpha, \beta, \gamma,$ &c. must satisfy the equations to the planes; hence, we have

$$l\lambda + m\mu + n\nu = \frac{1}{a}(lx + my + nz),$$

$$l'\lambda + m'\mu + n'\nu = \frac{1}{b}(l'x + m'y + n'z),$$

$$l''\lambda + m''\mu + n''\nu = \frac{1}{c}(l''x + m''y + n''z).$$

From these we can determine $\lambda, \mu, \nu,$ in the form

$$\lambda = \frac{f}{a}(lx + my + nz), \quad \mu = \frac{g}{b}(l'x + m'y + n'z), \quad \nu = \frac{h}{c}(l''x + m''y + n''z),$$

f, g, h, being functions of $l, m, n, l', m', n', l'', m'', n''$, of which the form is obvious. Hence, observing that $\lambda^2 + \mu^2 + \nu^2 = 1$, we have

$$\frac{f^2}{a^2}(lx + my + nz)^2 + \frac{g^2}{b^2}(l'x + m'y + n'z)^2 + \frac{h^2}{c^2}(l''x + m''y + n''z)^2 = 1,$$

as the equation to the required locus, which is evidently a central surface of the second degree.

PROB. 9. If through a fixed point O any three chords AA', BB', CC', be drawn in a surface of the second order, the locus of the intersection of the plane passing through A, B, C, with that passing through A', B', C', is a plane.

Take O as the origin, OA, OB, OC, as the axes of x, y, z; the equation to the surface being

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy + 2Cx + 2C'y + 2C''z + E = 0 \quad \ldots\ldots(1).$$

Now, if $OA = a$, $OB = b$, $OC = c$, $OA = a'$, $OB = b'$, $OC = c'$; the equation to the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

and that to $A'B'C'$ is

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1.$$

When the planes intersect we may combine the equations in any way we choose: adding them, we have

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) + z\left(\frac{1}{c} + \frac{1}{c'}\right) = 2.$$

This is a relation between the co-ordinates of the line of intersection, and it may also be considered as the equation to a plane in which that line lies. We have now to shew that it remains fixed in position when the position of the chords is changed, the point O remaining the same. For this purpose we observe that a and a', being the intercepts on the axis of x between the origin and the surface, are the roots of the equation

$$Ax^2 + 2Cx + E = 0,$$

derived from the equation to the surface by making $y = 0$, $z = 0$.

Hence, by the known relations between the roots of an equation and its coefficients,

$$\frac{1}{a}+\frac{1}{a'}=-\frac{2C}{E}.$$

In like manner we find

$$\frac{1}{b}+\frac{1}{b'}=-\frac{2C'}{E}, \quad \frac{1}{c}+\frac{1}{c'}=-\frac{2C''}{E}.$$

Substituting these values in the equation to the plane, it becomes

$$Cx+C'y+C''z+E=0 \;\ldots\ldots\; (2).$$

Now, if we were to change the position of the chords passing through O, we should in fact be simply changing the direction of the co-ordinates without altering the origin. The substitutions for effecting this transformation are linear, or of the form

$$x=ax'+by'+cz';$$

and therefore the groups of the terms of the first and second degrees in equation (1) will change independently of each other, the constant term not being altered. Consequently the equation (2), which is the same as the last four terms of (1), will experience the same change from the transformation of co-ordinates as it would do if it were deduced from (1) after the transformation had been then effected. Consequently the position of the plane (2) remains the same when the co-ordinate axes are changed, or it is the locus of the lines of intersection of the planes passing through the extremities of the chords.

Prob. 10. If there be two homofocal ellipsoids,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1,$$

and we take in one two points, $P(x, y, z)$ and $Q(\xi, \eta, \zeta)$, and in the other two points, $P'(x', y', z')$ and $Q'(\xi', \eta', \zeta')$, so connected, that

$$\frac{x}{x'}=\frac{a}{a'}=\frac{\xi}{\xi'}, \qquad \frac{y}{y'}=\frac{b}{b'}=\frac{\eta}{\eta'}, \qquad \frac{z}{z'}=\frac{c}{c'}=\frac{\zeta}{\zeta'},$$

then shall $PQ'=P'Q$,

$$(PQ')^2=(\xi'-x)^2+(\eta'-y)^2+(\zeta'-z)^2,$$
$$(P'Q)^2=(\xi-x')^2+(\eta-y')^2+(\zeta-z')^2.$$

Eliminating x', y', z', ξ', η', ζ', by means of the preceding relations,

$$(PQ')^2 = \left(\xi\frac{a'}{a} - x\right)^2 + \left(\eta\frac{b'}{b} - y\right)^2 + \left(\zeta\frac{c'}{c} - z\right)^2,$$

$$(P'Q)^2 = \left(\xi - x\frac{a'}{a}\right)^2 + \left(\eta - y\frac{b'}{b}\right)^2 + \left(\zeta - z\frac{c'}{c}\right)^2,$$

$$(P'Q)^2 - (PQ')^2 = \left(\frac{\xi^2}{a^2} - \frac{x^2}{a^2}\right)(a^2 - a'^2) + \left(\frac{\eta^2}{b^2} - \frac{y^2}{b^2}\right)(b^2 - b'^2) + \left(\frac{\zeta^2}{c^2} - \frac{z^2}{c^2}\right)(c^2 - c'^2).$$

But, as the surfaces are homofocal,

$$a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2;$$

hence

$$(P'Q)^2 - (PQ')^2 = (a^2 - a'^2)\left\{\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right) - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\right\}.$$

Since x, y, z, ξ, η, ζ, are co-ordinates of the ellipse, the second side vanishes, and we have

$$(P'Q)^2 - (PQ')^2 = 0, \quad \text{or} \quad P'Q = PQ'.$$

PROB. 11. To find the locus of the middle points of all the chords in a central surface of the second order, which pass through a given point.

Let $$Ax^2 + By^2 + Cz^2 = D \quad \ldots\ldots\ldots (1)$$

be the equation to the surface, and

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \ldots\ldots\ldots (2),$$

the equation to any chord passing through the fixed point a, b, c. Then, if x_1, y_1, z_1, x_2, y_2, z_2, be the co-ordinates of the points where the chords meet the surface, we have, from (2),

$$\frac{x_1 - x_2}{l} = \frac{y_1 - y_2}{m} = \frac{z_1 - z_2}{n} \quad \ldots\ldots\ldots\ldots (3),$$

and, from (1), the two conditions

$$Ax_1^2 + By_1^2 + Cz_1^2 = D, \quad Ax_2^2 + By_2^2 + Cz_2^2 = D;$$

whence $$A(x_1^2 - x_2^2) + B(y_1^2 - y_2^2) + C(z_1^2 - z_2^2) = 0.$$

Dividing each term of this last equation by the corresponding members of (3), we have

$$Al(x_1 + x_2) + Bm(y_1 + y_2) + Cn(z_1 + z_2) = 0.$$

Now if x', y', z', be the co-ordinates of the middle point of the chord, $x' = \frac{1}{2}(x_1 + x_2)$, $y' = \frac{1}{2}(y_1 + y_2)$, $z' = \frac{1}{2}(z_1 + z_2)$,

and the last equation becomes

$$Alx' + Bmy' + Cnz' = 0.$$

But x', y', z', satisfy equations (2), so that

$$\frac{x' - a}{l} = \frac{y' - b}{m} = \frac{z' - c}{n}.$$

Hence, eliminating l, m, n, between the last two equations, we find $\quad Ax'(x' - a) + By'(y' - b) + Cz'(z' - c) = 0,$

as the equation to the required locus, which is evidently a surface similar to the original one, and passing through the origin and the fixed point.

PROB. 12. To find the equation to the surface which is described in the following manner:—

At the middle point of every central plane section of an ellipsoid a normal to the plane is drawn, and along this, in the same direction, are measured lines equal to the principal axes of the section. The extremities of these lines will trace out a surface of two sheets, which is the one in question.

By Art. (122) the principal axes of a section of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

made by a plane $lx + my + nz = 0$, are given by the equation

$$\frac{a^2l^2}{r^2 - a^2} + \frac{b^2m^2}{r^2 - b^2} + \frac{c^2n^2}{r^2 - c^2} = 0 \;\ldots\ldots\ldots\; (1).$$

The equations to a normal to the plane through the origin are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = \rho \;\ldots\ldots\ldots\ldots\ldots\; (2);$$

and if along this normal we measure distances equal to the principal axes of the section, ρ, being the length of the distance, must be a root of the preceding quadratic: or we may take it

as equal to r: eliminating, then, l, m, n, between (1) and (2), we have

$$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0,$$

as the equation to the surface. If we put for r^2 its value $x^2 + y^2 + z^2$, and get rid of the denominators, the equation to the surface becomes

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2 + a^2b^2c^2 = 0.$$

This is Fresnel's construction for the Wave Surface in the Theory of Light.

THE END.

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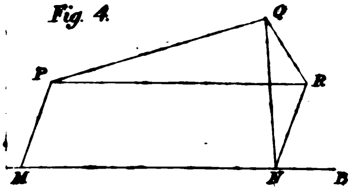
Fig. 4.



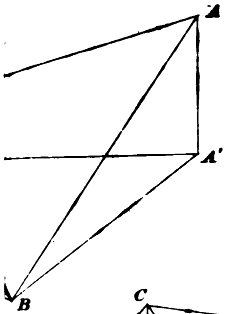


Fig 10.

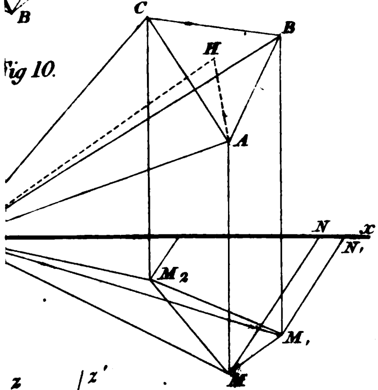


Fig. 14.

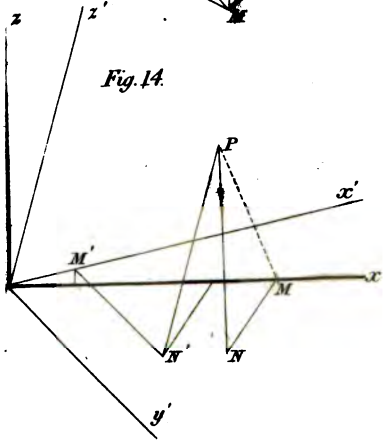


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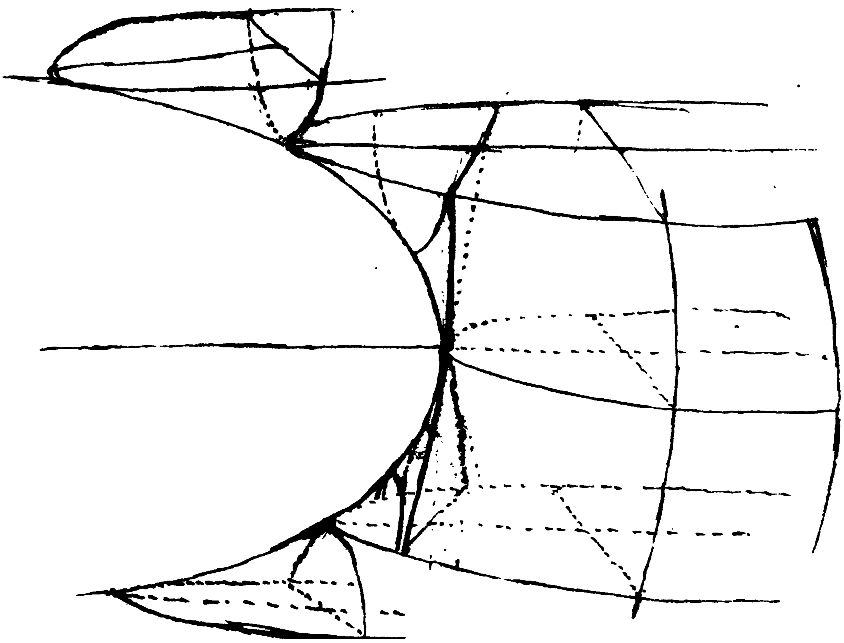
Fig 23.



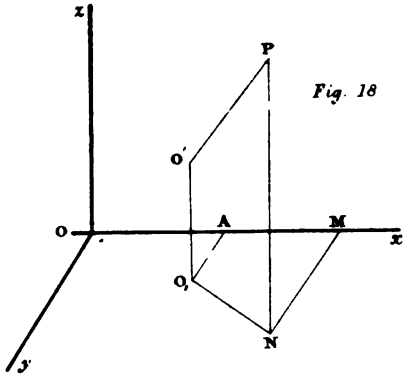
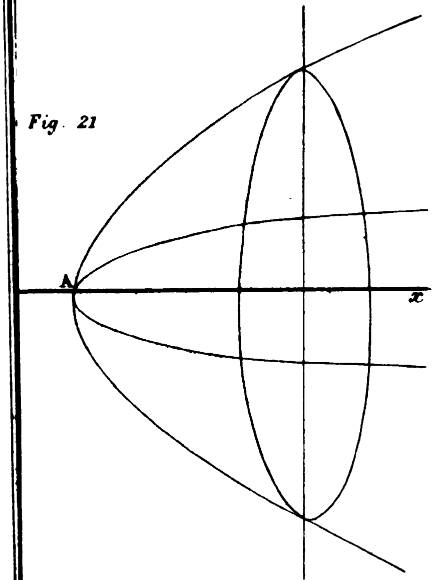


Fig. 18

Fig. 21



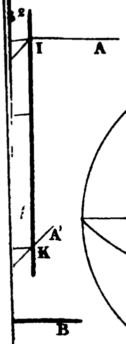
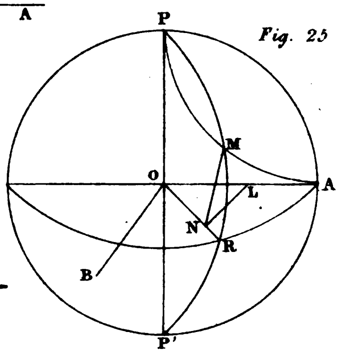


Fig. 25



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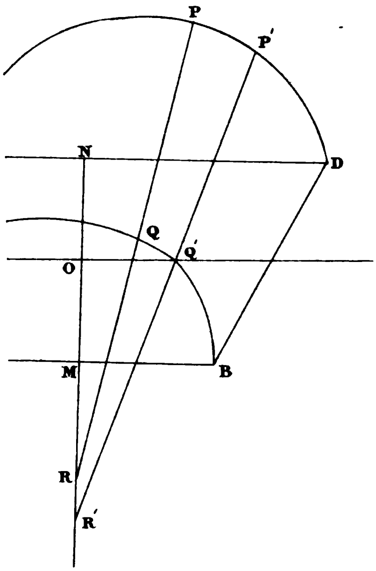
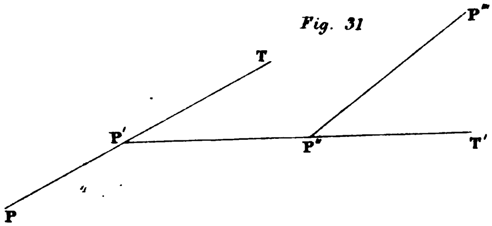
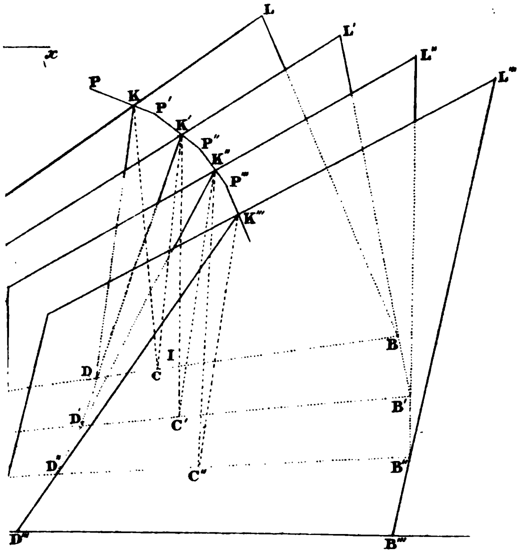


Fig. 31





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